

## Linear and nonlinear vibrations of inhomogeneous Euler-Bernoulli beam

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**Abstract.** Dynamic problems arising from the Euler-Bernoulli beam model with inhomogeneous elastic properties are considered. The method of Green's function and perturbation theory are employed to find the deflection in the beam correct to the first-order. Eigenvalue problems appearing from transverse vibrations of inhomogeneous beams in linear and nonlinear cases are also discussed.

**Keywords:** vibration; inhomogeneous; Euler-Bernoulli beam

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### 1. Introduction

Beams and girders are extensively used in civil and mechanical engineering. One of the earliest models was Euler-Bernoulli model that was used to study the bending of beams. The derivation of Euler-Bernoulli beam equation has been given by Duque (2015). The model was based on small deflections of a beam subjected to lateral loads. We may refer to Truesdell (1983) for an account of development of this approach in 1750. The study of beam equation is quite important in a number of engineering situations, see for instance Ebrahimi and Barati (2018), Rizov (2017), Huang *et al.* (2017), Nejad *et al.* (2017), Mohammadimehr and Alimirzaei (2016), and Webb *et al.* (2008). It was of a little consequence in terms of applications till it became a cornerstone of engineering in the late 19th century. Han *et al.* (1999) provided a good description of different models of elastic beams including the Euler-Bernoulli beam. Gupta (1988) proved the existence and uniqueness of solution to the fourth-order equation arising from bending of an elastic beam. Abu-Hilal (2003) studied forced vibration of Euler-Bernoulli beam in the case of different homogenous and elastic boundary conditions for dynamic response due to distributed or concentrated loads.

The use of spectral properties, Green's function and perturbation method has been an important tool in second-order problems arising from vibration, elastic, acoustic and electromagnetic waves. Discussion on these methods may be found in Lindell and Olyslager (2001), Logan (2007), and Stakgold and Holst (2011). Stuwe and Werner (1996) used Green's function to study potential flow

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(linear case) in infinite cylindrical channels. Graef and Yang (1999) discussed the latter problem for non-linear load and established existence results advantaging the Green function approach. Graef *et al.* (2009) obtained positive solutions of nonlinear fourth-order problems using this approach. More recently, Pietramala (2011) used the Green function to study the beam equation with nonlinear boundary conditions. Furthermore, Teterina (2013) obtained the relevant results for the Green function of the fourth-order operator and used it to solve some boundary value problems. Morrison (2007) in his Master's thesis has applied Green's function to solve third-order nonlinear boundary value problems. Green's function has also been used in the frequency analysis of axially loaded stepped beams and the frequency response analysis of beams and plane frames with the inclusion of viscoelastic damping, see Kukla and Zamojska (2007), Burlon *et al.* (2016), and Failla (2016). Li, *et al.* (2014) discussed Green's functions of the forced vibration of Timoshenko beams with damping effect. Many authors studied the inhomogeneity in the medium using perturbation approach. For instance, Ghosh (1970) employed the latter approach to investigate Love waves in an inhomogeneous medium. Subsequently, Zaman *et al.* (1990) and Asghar *et al.* (1991) adopted the Green function and perturbation method to discuss the field due to a point source in an inhomogeneous medium, dispersion of Love waves in a stochastic layer (Zaman and Al-Zayer 2004) and inverse scattering in multi-layer inverse problem (Zaman *et al.* 2006). Orucoglu (2005) has also valued the Green function approach to deal with a completely inhomogeneous boundary value problem.

In this paper, we discuss the dynamic problems arising from some beam equations. The beam is considered to be inhomogeneous. This means that the elastic properties of the beam material change with the space variable. The focus of this study is to apply analytical approach based upon perturbation method and Green's function.

## 2. Eigenvalue perturbation

Let  $L(\varepsilon)$  be a self-adjoint operator defined in a real Hilbert Space  $H$ . Assume that  $L(\varepsilon)$  depends continuously on  $\varepsilon$ , i.e.,

$$\lim_{\Delta\varepsilon \rightarrow 0} \|L(\varepsilon + \Delta\varepsilon) - L(\varepsilon)\| = 0 \quad (2.1)$$

We relate the spectrum of the perturbed operator  $L(\varepsilon)$  to the presumable known spectrum of the base operator  $L = L(0)$ .

**Theorem 2.1.** *Let  $\lambda_n \neq 0$  be simple eigenvalues of the operator  $L$  with normalized eigenfunctions  $e_n$ . Then in some neighborhood of  $\varepsilon = 0$ , there exists an eigenpair  $(\lambda(\varepsilon); e(\varepsilon))$  of Eq. (2.1) with the following properties*

$$\lim_{\varepsilon \rightarrow 0} \lambda(\varepsilon) = \lambda_n \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} e(\varepsilon) = e_n.$$

Before discussing the fourth-order equation, we give the following illustrative example. Consider the boundary value problem

$$\begin{aligned} -u''(x, \varepsilon) + (1 + \varepsilon x^2)u(x, \varepsilon) &= \mu u(x, \varepsilon), & 0 < x < 1; \\ u(0, \varepsilon) &= 0, & u(1, \varepsilon) &= 0. \end{aligned} \quad (2.2)$$

When  $\varepsilon = 0$ , we have

$$\begin{aligned} -u''(x, 0) &= (\mu - 1)u(x, \varepsilon), \quad 0 < x < 1; \\ u(0, 0) &= 0, \quad u(1, 0) = 0. \end{aligned} \quad (2.3)$$

The eigenvalues and corresponding normalized eigenfunctions of (2.3) are given by

$$\begin{aligned} \mu_n &= n^2\mu^2 + 1, \quad n = 1, 2, \dots \\ e_n(x) &= \sqrt{2} \sin(n\pi x), \quad n = 1, 2, \dots, \end{aligned} \quad (2.4)$$

respectively. From Theorem 2.1, for the boundary value problem (2.2), there is a branch  $(\mu(\varepsilon), u(x, \varepsilon)) \rightarrow (\mu_n, e_n(x))$  as  $\varepsilon \rightarrow 0$ . As the operator in (2.2) depends continuously on  $\varepsilon$ , we can differentiate (2.2) with respect to  $\varepsilon$  to obtain the following

$$\begin{aligned} -u_\varepsilon''(x, \varepsilon) + (1 + \varepsilon x^2)u_\varepsilon(x, \varepsilon) - \mu u_\varepsilon(x, \varepsilon) &= \mu_\varepsilon u(x, \varepsilon) - x^2 u(x, \varepsilon); \\ u_\varepsilon(0, \varepsilon) &= 0. \end{aligned} \quad (2.5)$$

Since  $\varepsilon = 0$ , we get an inhomogeneous equation for  $u_\varepsilon(x, \varepsilon)$  as follows

$$-u_\varepsilon''(x, \varepsilon) + u_\varepsilon(x, \varepsilon) - \mu_n u_\varepsilon(x, \varepsilon) = \mu_\varepsilon(0)e_n(x) - x^2 e_n(x). \quad (2.6)$$

Eq. (2.6) cannot be solved for  $u_\varepsilon(x, \varepsilon)$  unless  $\mu_\varepsilon(0)$  is known and it must satisfy the following solvability condition

$$\langle \mu_\varepsilon(0)e_n(x) - x^2 e_n(x), e_n(x) \rangle = 0. \quad (2.7)$$

Eq. (2.7) leads to

$$\mu_\varepsilon(0) = \int_0^1 x^2 e_n^2(x) dx \quad (2.8)$$

and so, we have

$$\mu(\varepsilon) = \mu_n + \varepsilon \int_0^1 x^2 e_n^2(x) dx + \dots, \quad (2.9)$$

See Stakgold and Holst (2011). Furthermore, there exists a unique solution  $w(x)$  that is orthogonal to the eigenfunction  $e_n(x)$  given by

$$w(x) = - \sum_{m \neq n} \frac{\langle x^2 e_n(x), e_m(x) \rangle}{(m^2 - n^2)\pi^2} e_m(x),$$

where  $e_n(x) = \sqrt{2} \sin(n\pi x)$ . Thus, using the normalization  $\langle u_n, e_n \rangle = 1$  we obtain

$$u_n(x, \varepsilon) = e_n(x) + \varepsilon w(x).$$

### 3. Transverse vibration of beam as an eigenvalue problem

Consider the following problem

$$\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 u(x, t)}{\partial x^2} \right) = \rho \frac{\partial^2 u(x, t)}{\partial t^2}, \quad 0 < x < 1, \quad t > 0. \quad (3.1)$$

When  $EI = q_0$  (constant), we have

$$\frac{\partial^4 u(x, t)}{\partial x^4} = \frac{\rho}{q_0} \frac{\partial^2 u(x, t)}{\partial t^2}, \quad q_0 \neq 0. \quad (3.2)$$

Assume that the vibrations are time harmonic, i.e.,

$$u(x, t) = u(x)e^{i\omega t}, \quad (3.3)$$

where,  $\omega$  is the angular frequency. Substituting (3.3) to (3.2) yields

$$\frac{d^4 u(x)}{dx^4} = -\frac{\rho\omega^2}{q_0} u(x),$$

or

$$\frac{d^4 u(x)}{dx^4} + \lambda u(x) = 0, \quad 0 < x < 1, \quad (3.4)$$

where  $\lambda = \rho\omega^2/q_0$ . Thus, the transverse vibration of an elastic Euler-Bernoulli beam can be cast into an equivalent eigenvalue problem of the type given in Eq. (3.4).

In the following examples, we solve the eigenvalue problem (3.4) in the following three cases: beam with hinged, with clamped and clamped-free boundary conditions.

Example 3.1 (Beam with Both Ends Hinged). In this case, we have the following eigenvalue problem

$$\begin{cases} \frac{d^4 u}{dx^4} + \lambda u = 0, & 0 < x < 1; \\ \text{with hinged boundary condition} \\ u(0) = 0, u''(0) = 0, u(1) = 0, u''(1) = 0. \end{cases} \quad (3.5)$$

Observe that  $\lambda = 0$  is not an eigenvalue of the problem, which gives the trivial solution  $u = 0$ . Since all eigenvalues are real and positive, we assume that  $\lambda = \alpha^4$  for some  $\alpha \in \mathbb{R}$ . The characteristic equation is given by

$$m^4 + \alpha^4 = 0.$$

We have

$$\begin{aligned} m_1 &= \frac{\alpha}{\sqrt{2}}(1 + i), & m_2 &= \frac{\alpha}{\sqrt{2}}(-1 + i), \\ m_3 &= \frac{\alpha}{\sqrt{2}}(-1 - i), & m_4 &= \frac{\alpha}{\sqrt{2}}(1 - i). \end{aligned} \quad (3.6)$$

The general solution of (3.5) is

$$u(x) = e^{\alpha x/\sqrt{2}} \left[ c_1 \cos\left(\frac{\alpha x}{\sqrt{2}}\right) + c_2 \sin\left(\frac{\alpha x}{\sqrt{2}}\right) \right] + e^{-\alpha x/\sqrt{2}} \left[ c_3 \cos\left(\frac{\alpha x}{\sqrt{2}}\right) + c_4 \sin\left(\frac{\alpha x}{\sqrt{2}}\right) \right].$$

Using the given boundary conditions, we obtain

$$\lambda_n = \alpha_n^4 = 4n^4\pi^4, \quad n = 1, 2, \dots,$$

which are the eigenvalues of the supply-supported beam (hinged beam with both ends). Hence, the corresponding normalized eigenfunctions are

$$u_n(x) = \sqrt{2} \sin(n\pi x), \quad n = 1, 2, \dots$$

Example 3.2 (Beam with Both Ends Clamped). In this case, we have the following eigenvalue problem

$$\begin{cases} \frac{d^4 u}{dx^4} + \lambda u = 0, & 0 < x < 1; \\ \text{with clamped boundary condition} \\ u(0) = 0, u'(0) = 0, u(1) = 0, u'(1) = 0. \end{cases} \quad (3.7)$$

We can easily check that  $\lambda = 0$  is not an eigenvalue of the problem, which gives the trivial solution  $u = 0$ . As all eigenvalues are real and positive, we assume that  $\lambda = \beta^4$  for some  $\beta \in \mathbb{R}$ . The characteristic equation is given by

$$m^4 + \beta^4 = 0.$$

We obtain the general solution of (3.7)

$$u(x) = e^{\beta x/\sqrt{2}} \left[ c_1 \cos\left(\frac{\beta x}{\sqrt{2}}\right) + c_2 \sin\left(\frac{\beta x}{\sqrt{2}}\right) \right] + e^{-\beta x/\sqrt{2}} \left[ c_3 \cos\left(\frac{\beta x}{\sqrt{2}}\right) + c_4 \sin\left(\frac{\beta x}{\sqrt{2}}\right) \right].$$

Imposing the given boundary conditions, we obtain the smallest positive value

$$\lambda = \beta^4 = (4.7300407)^4 = 500.56655.$$

Example 3.3 (Beam with Ends Clamped-Free). In this case, we have the following eigenvalue problem

$$\begin{cases} \frac{d^4 u}{dx^4} + \lambda u = 0, & 0 < x < 1; \\ \text{with clamped – free boundary conditions,} \\ u(0) = 0, u'(0) = 0, u''(1) = 0, u'''(1) = 0. \end{cases} \quad (3.8)$$

Observe that  $\lambda = 0$  is not an eigenvalue of the problem, which gives the trivial solution  $u = 0$ . Since all eigenvalues are real and positive, we may assume that  $\lambda = \gamma^4$  for some  $\gamma \in \mathbb{R}$ . The characteristic equation is given by

$$m^4 + \gamma^4 = 0.$$

The general solution of (3.8) is

$$u(x) = e^{\gamma x/\sqrt{2}} \left[ c_1 \cos\left(\frac{\gamma x}{\sqrt{2}}\right) + c_2 \sin\left(\frac{\gamma x}{\sqrt{2}}\right) \right] + e^{-\gamma x/\sqrt{2}} \left[ c_3 \cos\left(\frac{\gamma x}{\sqrt{2}}\right) + c_4 \sin\left(\frac{\gamma x}{\sqrt{2}}\right) \right].$$

Imposing the given boundary conditions, we obtain the smallest positive value

$$\lambda = \gamma^4 = (1.8751)^4 = 12.3623.$$

#### 4. Vibration of inhomogeneous beams

In this section, we consider the free vibration of inhomogeneous Euler-Bernoulli Beam. We assume that the variation of elastic properties is small, so that we can write

$$EI(x) = q(x) = q_0 + \varepsilon q_1(x) + \dots, \quad (4.1)$$

where  $q_0$  is a nonzero constant and  $q_1(x)$  is a differentiable function of  $x$ . The equation of the vibrating beam can be written as

$$\frac{d^2}{dx^2} \left[ (q_0 + \varepsilon q_1(x)) \frac{d^2 u(x, \varepsilon)}{dx^2} + \dots \right] = \lambda(\varepsilon) u(x, \varepsilon), \quad 0 < x < 1; \quad (4.2)$$

with appropriate boundary conditions.

Suppose that the beam is hinged at both ends. We have the conditions

$$u(0) = 0, u''(0) = 0, u(1) = 0, u''(1) = 0. \quad (4.3)$$

Note that for  $\varepsilon = 0$ , Eq. (4.2) is reduced to the one studied in Section 3. The eigenvalues are  $\lambda_n = 4n^4\pi^4$ ,  $n = 1, 2, \dots$  with corresponding normalized eigenfunctions

$$e_n = \sqrt{2} \sin(n\pi x), \quad n = 1, 2, \dots$$

Eq. (4.2) can be written as

$$u^{(4)}(x, \varepsilon) + \varepsilon \left[ \frac{q_1(x)}{q_0} u^{(4)}(x, \varepsilon) + 2 \frac{q_1'(x)}{q_0} u'''(x, \varepsilon) + \frac{q_1''(x)}{q_0} u''(x, \varepsilon) \right] + \dots = \lambda(\varepsilon) u(x, \varepsilon), \quad (4.4)$$

$$0 < x < 1$$

with hinged boundary conditions

$$u(0, \varepsilon) = 0, u''(0, \varepsilon) = 0, u(1, \varepsilon) = 0, u''(1, \varepsilon) = 0.$$

Assume that we can differentiate both sides of Equation (4.4) with respect to  $\varepsilon$ . We obtain

$$\begin{aligned} u_\varepsilon^{(4)}(x, \varepsilon) + \left[ \frac{q_1(x)}{q_0} u^{(4)}(x, \varepsilon) + 2 \frac{q_1'(x)}{q_0} u'''(x, \varepsilon) + \frac{q_1''(x)}{q_0} u''(x, \varepsilon) \right] \\ + \varepsilon \left[ \frac{q_1(x)}{q_0} u_\varepsilon^{(4)}(x, \varepsilon) + 2 \frac{q_1'(x)}{q_0} u_\varepsilon'''(x, \varepsilon) + \frac{q_1''(x)}{q_0} u_\varepsilon''(x, \varepsilon) \right] \\ + \dots = \lambda_\varepsilon(\varepsilon) u(x, \varepsilon) + \lambda(\varepsilon) u_\varepsilon(x, \varepsilon). \end{aligned} \quad (4.5)$$

The eigenvalues  $\lambda(\varepsilon)$  and normalized eigenfunctions  $u(x, \varepsilon)$  for  $\varepsilon \rightarrow 0$  are  $\lambda(0) = \lambda_n = 4n^4\pi^4$  and  $u(x, 0) = u_n(x) = \sqrt{2} \sin(n\pi x)$  respectively, where  $n = 1, 2, \dots$ . Thus, Eq. (4.5) reads

$$\begin{aligned} u_\varepsilon^{(4)}(x, 0) + \lambda_n u_\varepsilon(x, 0) = \sqrt{2} \lambda_\varepsilon(0) \sin(n\pi x) - 4\sqrt{2} n^4 \pi^4 \frac{q_1(x)}{q_0} \sin(n\pi x) \\ + 8n^3 \pi^3 \frac{q_1'(x)}{q_0} \cos(n\pi x) + 2\sqrt{2} n^2 \pi^2 \frac{q_1''(x)}{q_0} \sin(n\pi x), \end{aligned} \quad (4.6)$$

with boundary conditions

$$u_\varepsilon(0,0) = 0, u_\varepsilon''(0,0) = 0, u_\varepsilon(1,0) = 0, u_\varepsilon''(1,0) = 0. \quad (4.7)$$

Since  $\lambda_\varepsilon(0)$  is unknown, we cannot solve (4.6) directly. The corresponding homogenous problem has the non-trivial solution  $e_n(x)$ . Thus, Eq. (4.6) with boundary conditions in (4.7) has a unique solution if the following solvability condition is satisfied

$$\begin{aligned} & \left( \sqrt{2}\lambda_\varepsilon(0) \sin(n\pi x) \right. \\ & - \left( 4\sqrt{2} n^4 \pi^4 \frac{q_1(x)}{q_0} \sin(n\pi x) - 8n^3 \pi^3 \frac{q_1'(x)}{q_0} \cos(n\pi x) \right. \\ & \left. \left. - 2\sqrt{2} n^2 \pi^2 \frac{q_1''(x)}{q_0} \sin(n\pi x) \right), \sqrt{2} \sin(n\pi x) \right) = 0. \end{aligned} \quad (4.8)$$

We obtain

$$\begin{aligned} 2\lambda_\varepsilon(0) \int_0^1 \sin^2(n\pi x) dx - \frac{4n^2 \pi^2}{q_0} \int_0^1 (2n^2 \pi^2 q_1(x) \sin(n\pi x) - 2\sqrt{2} n \pi q_1'(x) \cos(n\pi x) \\ - q_1''(x) \sin(n\pi x)) \sin(n\pi x) dx = 0. \end{aligned} \quad (4.9)$$

This gives

$$\begin{aligned} \lambda_\varepsilon(0) = \frac{2n^2 \pi^2}{q_0} \int_0^1 (2n^2 \pi^2 q_1(x) \sin^2(n\pi x) - 2\sqrt{2} n \pi q_1'(x) \sin(2n\pi x) \\ - q_1''(x) \sin^2(n\pi x)) dx \end{aligned} \quad (4.10)$$

Therefore, the general form of the eigenvalue for the vibrating inhomogeneous beam with hinged boundary conditions is

$$\lambda(\varepsilon) = \lambda_\varepsilon + \varepsilon \lambda_\varepsilon(0) + \dots$$

Hence,

$$\begin{aligned} \lambda(\varepsilon) = 4n^4 \pi^4 + \varepsilon \frac{2n^2 \pi^2}{q_0} \int_0^1 [2n^2 \pi^2 q_1(x) \sin^2(n\pi x) - 2\sqrt{2} n \pi q_1'(x) \sin(2n\pi x) \\ - q_1''(x) \sin^2(n\pi x)] dx + \dots \end{aligned} \quad (4.11)$$

The smallest eigenvalue is when  $n = 1$

$$\begin{aligned} \lambda(\varepsilon) = 4\pi^4 + \varepsilon \frac{2\pi^2}{q_0} \int_0^1 [2\pi^2 q_1(x) \sin^2(\pi x) - 2\sqrt{2} \pi q_1'(x) \sin(2\pi x) \\ - q_1''(x) \sin^2(\pi x)] dx + \dots \end{aligned} \quad (4.12)$$

Eq. (4.6) with the given hinged boundary conditions can be solved for  $u_\varepsilon(x, 0)$ . The unique solution  $w(x)$  that is orthogonal to  $e_n(x)$  is given by

$$w(x) = \sum_{m \neq n} \left( \frac{-2 \sin(m\pi x)}{(m^4 - n^4)\pi^4} \left( n^4 \pi^4 \frac{q_1(x)}{q_0} \sin(n\pi x) - 2n^3 \pi^3 \frac{q_1'(x)}{q_0} \cos(n\pi x) - n^2 \pi^2 \frac{q_1''(x)}{q_0} - n^2 \pi^2 \frac{q_1''(x)}{q_0} \sin(n\pi x), \sqrt{2} \sin(m\pi x) \right) \right), \quad (4.13)$$

compare Stakgold and Holst (2011).

In Table 1, we present eigenvalues of the perturbed problem (concrete beam) with hinged both ends. Table 2 gives eigenvalues of the perturbed problem (steel beam) with hinged both ends. Note that in each case, we take  $q_1(x) = 10^{18}x$  as a linear variation.

Table 1 Eigenvalues of vibrating inhomogeneous hinged concrete beam

| Values of $\varepsilon$ | Eigenvalue $\lambda(\varepsilon)$ |
|-------------------------|-----------------------------------|
| $\varepsilon = 0$       | $\pi^4$                           |
| $\varepsilon = 0.1$     | $1.3719 \times 10^{10}$           |
| $\varepsilon = 0.2$     | $2.75037 \times 10^{10}$          |
| $\varepsilon = 0.3$     | $4.12556 \times 10^{10}$          |

Table 2 Eigenvalues of vibrating inhomogeneous hinged steel beam

| Values of $\varepsilon$ | Eigenvalue $\lambda(\varepsilon)$ |
|-------------------------|-----------------------------------|
| $\varepsilon = 0$       | $\pi^4$                           |
| $\varepsilon = 0.1$     | $1.11325 \times 10^9$             |
| $\varepsilon = 0.2$     | $2.22649 \times 10^9$             |
| $\varepsilon = 0.3$     | $3.33974 \times 10^9$             |

#### 4.1 An unperturbed nonlinear problem

In this subsection, we apply the Green's function obtained from the fourth-order operator  $d^4/dx^4$  to a non-linear eigenvalue problem.

Consider the following problem

$$\begin{cases} \frac{d^4 u(x)}{dx^4} = \alpha \sin(u(x)), & 0 < x < 1; \\ \text{with hinged boundary conditions} \\ u(0) = 0, u''(0) = 0, u(1) = 0, u''(1) = 0. \end{cases} \quad (4.14)$$

For  $\alpha = 0$ , the translation of (4.14) of an interareal equation is simple, because (4.14) has a non-trivial solution. Note that (4.14) has the trivial solution  $u = 0$ . The linearization of (4.14) about  $u = 0$  is given by

$$\begin{cases} \frac{d^4 u(x)}{dx^4} = \alpha u(x), & 0 < x < 1; \\ \text{with hinged boundary conditions} \\ u(0) = 0, u''(0) = 0, u(1) = 0, u''(1) = 0. \end{cases} \quad (4.15)$$



In fact, we have found the eigenvalues and normalized eigenfunctions for (4.15) in the previous section given by  $\alpha_n = 4n^4\pi^4$ ,  $e_n(x) = \sqrt{2} \sin(n\pi x)$ ,  $n = 1, 2, \dots$  respectively. Since zero is not an eigenvalue of (4.15), we can construct Green's function  $G(x, \xi)$  satisfying

$$\begin{cases} \frac{d^4 G(x, \xi)}{dx^4} = \delta(x - \xi), & x \neq \xi, & 0 < x < 1; \\ \text{with hinged boundary conditions} \\ G(0, \xi) = 0, G''(0, \xi) = 0, G(1, \xi) = 0, G''(1, \xi) = 0. \end{cases} \quad (4.16)$$

The Green's function of (4.16) is

$$G(x, \xi) = \begin{cases} \frac{1}{6}\xi(1-\xi)x + \frac{1}{6}\xi(1-\xi)x^3, & 0 \leq x < \xi \leq 1; \\ \frac{1}{3}\xi(1-\xi)(1-x) + \frac{1}{6}\xi(1-x)^3, & 0 \leq \xi < x \leq 1. \end{cases} \quad (4.17)$$

Now Problem (4.14) can be written as the following equivalent integral equation

$$\lambda u(x) = \int_0^1 G(x, \xi) \sin(u(\xi)) d\xi = Au(x) \quad (4.18)$$

where  $\lambda = 1/\alpha$  and  $A$  is a nonlinear Hammerstein integral operator whose linearization at  $u = 0$  is a linear operator  $B$  with kernel  $G(x, \xi)$ . Hence, the problem

$$Bu(x) = \lambda u(x) \quad (4.19)$$

which can be written as

$$\lambda u(x) = \int_0^1 G(x, \xi) u(\xi) d\xi,$$

has eigenvalues  $\lambda_n = 1/\alpha_n$ ,  $n = 1, 2, \dots$  and eigenfunctions  $e_n(x) = \sqrt{2} \sin(n\pi x)$ ,  $n = 1, 2, \dots$ . Taking norm of Eq. (4.19) gives

$$|\lambda| \|u(x)\| = \|\lambda u(x)\| = \|Bu(x)\| \leq \|B\| \|u(x)\|.$$

Thus,  $\|B\|$  is the largest eigenvalue  $\lambda_1 = 1/4\pi^4$  of the linear operator  $B$ . Therefore, Problem (4.18) can have a non-trivial solution if  $0 < \lambda \leq 1/(4\pi^4)$ , i.e.,  $\alpha \geq 4\pi^4$ .

#### 4.2 A Perturbed nonlinear problem

Consider the following problem

$$\begin{cases} \frac{d^4 u}{dx^4} + \varepsilon \frac{d^2}{dx^2} \left( \frac{q_1(x)}{q_0} \frac{d^2 u}{dx^2} \right) = \alpha \sin u, & 0 < x < 1; \\ \text{with hinged boundary conditions} \\ u(0) = 0, u''(0) = 0, u(1) = 0, u''(1) = 0. \end{cases} \quad (4.20)$$

We put a small perturbation on the left-hand side of the first equation in Problem (4.14). It is assumed that the variation of elastic properties is small. When  $\alpha = 0$ , the translation of (4.20) of an interareal equation is simple, because (4.20) has a non-trivial solution. Observe that (4.20)

possesses the trivial solution  $u = 0$ . The linearization of (4.20) about  $u = 0$  yields

$$\begin{cases} \frac{d^4 u}{dx^4} + \varepsilon \left[ \frac{q_1(x)}{q_0} \frac{d^4 u}{dx^4} + 2 \frac{q_1'(x)}{q_0} \frac{d^3 u}{dx^3} + \frac{q_1''(x)}{q_0} \frac{d^2 u}{dx^2} \right] = \alpha u, & 0 < x < 1; \\ \text{with hinged boundary conditions} \\ u(0) = 0, u''(0) = 0, u(1) = 0, u''(1) = 0. \end{cases} \quad (4.21)$$

In fact, we have found the eigenvalues and eigenfunctions for (4.21) in the previous section given by

$$\alpha_{linear} = 4n^2\pi^2 + \varepsilon\alpha_\varepsilon(0), \quad n = 1, 2, \dots$$

and

$$u_n(x, \varepsilon) = e_n(x) + w(x)$$

respectively, where  $\alpha_n(\varepsilon)$  and  $w(x)$  are written in (4.11) and (4.13). Since zero is not an eigenvalue of (4.21), we can construct Green's function  $G_\varepsilon(x, \varepsilon)$  satisfying

$$\begin{cases} \frac{d^4 G(x, \xi)}{dx^4} + \varepsilon \left[ \frac{q_1(x)}{q_0} \frac{d^4 G(x, \xi)}{dx^4} + 2 \frac{q_1'(x)}{q_0} \frac{d^3 G(x, \xi)}{dx^3} + \frac{q_1''(x)}{q_0} \frac{d^2 G(x, \xi)}{dx^2} \right] = \delta(x - \xi), \\ x \neq \xi, \quad 0 < x < 1; \text{ with hinged boundary conditions} \\ G(0, \xi) = 0, G''(0, \xi) = 0, G(1, \xi) = 0, G''(1, \xi) = 0. \end{cases} \quad (4.22)$$

The Green's function of (4.22) is

$$G_\varepsilon(x, \xi) = -\varepsilon \int_0^1 \left[ \frac{q_1(x)}{q_0} \frac{d^4 G(x, \xi)}{dx^4} + 2 \frac{q_1'(x)}{q_0} \frac{d^3 G(x, \xi)}{dx^3} + \frac{q_1''(x)}{q_0} \frac{d^2 G(x, \xi)}{dx^2} \right] G(x, \xi) d\xi, \quad (4.23)$$

where  $G(x, \xi)$  is the Green's function of

$$\frac{d^4 G(x, \xi)}{dx^4} = \delta(x - \xi), \quad x \neq \xi, \quad 0 < x < 1.$$

Problem (4.20) can be written as follows as an equivalent integral equation

$$\lambda u(x) = \int_0^1 G_\varepsilon(x, \xi) \sin(u(\xi)) d\xi = Au(x), \quad (4.24)$$

where  $\lambda_{NL} = 1/\alpha_{NL}$  and  $A$  is a nonlinear Hammerstein integral operator whose linearization at  $u = 0$  is a linear operator  $B$  with kernel  $G_\varepsilon(x, \xi)$ . The equation

$$Bu(x) = \lambda u(x) \quad (4.25)$$

can be written as

$$\lambda u(x) = \int_0^1 G_\varepsilon(x, \xi) \sin(u(\xi)) d\xi.$$

Problem (4.25) has the eigenvalues

$$\lambda_L = \frac{1}{\alpha_L} = \frac{1}{4n^4\pi^4 + \varepsilon\alpha_\varepsilon(0)}, \quad n = 1, 2, \dots$$

and the normalized eigenfunctions

$$u_n(x, \varepsilon) = e_n(x) + \varepsilon w(x),$$

where  $\alpha(\varepsilon)$  and  $w(x)$  are given in (4.11) and (4.13), respectively. Taking norm of both sides of (4.25) yields

$$|\lambda| \|u(x)\| = \|\lambda u(x)\| = \|Bu(x)\| \leq \|B\| \|u(x)\|.$$

Thus,  $\|B\|$  is the largest eigenvalue  $\lambda_L$ . We can conclude that (4.24) has a nontrivial solution only if

$$0 < \lambda_{NL} \leq \frac{1}{4\pi^4 + \varepsilon\alpha_\varepsilon(0)}$$

and

$$\alpha = \alpha_{NL} = \frac{1}{\lambda_{NL}} \Rightarrow 4\pi^4 + \varepsilon\alpha_\varepsilon(0) \leq \alpha.$$

As an illustrative example, let  $q_1(x) = 10^{18}x$  be a linear variation. In Tables 3 and 4, we present eigenvalues of perturbed nonlinear problem in cases of concrete and steel beams with hinged both ends, respectively.

Table 1 Eigenvalues of vibrating nonlinear concrete beam with hinged ends

| Values of $\varepsilon$ | Eigenvalue of Problem (4.20)<br>$\lambda = \lambda_{NL} \leq$ | Eigenvalue of Problem (4.24)<br>$\alpha = \alpha_{NL} \geq$ |
|-------------------------|---|---|
| $\varepsilon = 0$       | $1/(4\pi^4)$  | $4\pi^4$  |
| $\varepsilon = 0.1$     | $7.2892 \times 10^{-11}$                                      | $1.3719 \times 10^{10}$                                     |
| $\varepsilon = 0.2$     | $3.6359 \times 10^{-11}$                                      | $2.75037 \times 10^{10}$                                    |
| $\varepsilon = 0.3$     | $2.4239 \times 10^{-11}$                                      | $4.12556 \times 10^{10}$                                    |

Table 2 Eigenvalues of vibrating nonlinear steel beam with hinged ends

| Values of $\varepsilon$ | Eigenvalue of Problem (4.20)<br>$\lambda = \lambda_{NL} \leq$ | Eigenvalue of Problem (4.24)<br>$\alpha = \alpha_{NL} \geq$ |
|-------------------------|---|---|
| $\varepsilon = 0$       | $1/(4\pi^4)$  | $4\pi^4$  |
| $\varepsilon = 0.1$     |   | $8.9827 \times 10^{-10}$                                    |
| $\varepsilon = 0.2$     |   | $4.4914 \times 10^{-10}$                                    |
| $\varepsilon = 0.3$     |   | $2.9942 \times 10^{-10}$                                    |

## 5. Conclusions

In this study, we considered the fourth-order boundary value problem arising from the Euler-Bernoulli model of an elastic beam held under deferent supports. We described some basic spectral properties of the resulting fourth-order operator and constructed Green's function for three sets of boundary conditions. We also considered dynamic problems of transverse vibrations and obtained the eigenvalues and corresponding eigenfunctions.

This paper focused on the study of dynamic problems for Euler-Bernoulli beam with variable elastic properties. We used the perturbations formulation to discuss both elastic and dynamic models. The eigenvalue problem arising from transverse vibrations of inhomogeneous beams has been studied using perturbation approach. The smallest eigenvalue representing the eigenfrequency of the vibrating beam was obtained. A non-linear eigenvalue problem for inhomogeneous elastic properties of beam has been studied to find the variation in eigenfrequency due to inhomogeneity.

Three sets of boundary conditions were used to solve our fourth-order unperturbed and perturbed boundary value problems. More sets of boundary value problems can be considered. Also, non-homogeneous boundary conditions, such as a force being applied at one end, may be considered.

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