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Investigation of dynamic response of "bridge girder-telpher-load" crane system due to telpher motion

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Abstract. The moving load causes the occurrence of vibrations in civil engineering structures such as bridges, railway lines, bridge cranes and others. A novel engineering method for separation of the variables in the differential equation of the elastic line of Bernoulli-Euler beam has been developed. The method can be utilized in engineering structures, leading to "a beam under moving load model" with generalized boundary conditions. This method has been implemented for analytical study of the dynamic response of the metal structure of a single girder bridge crane due to the telpher movement along the bridge girder. The modeled system includes: a crane bridge girder; a telpher, moving with a constant horizontal velocity; a load, elastically fixed to the telpher. The forced vibrations with their own frequencies and with a forced frequency, due to the telpher movement, have been analyzed. The loading resulting from the telpher uniform movement along the bridge girder is cyclical, which is a prerequisite for nucleation and propagation of fatigue cracks. The concept of "dynamic coefficient" has been introduced, which is defined as a ratio of the dynamic deflection of the bridge girder due to forced vibrations, to the static one. This ratio has been compared with the known from the literature empirical dynamic coefficient, which is due to the telpher track unevenness. The introduced dynamic coefficient shows larger values and has to be taken into account for engineering calculations of the bridge crane metal structure. In order to verify the degree of approximation, the obtained results have been compared with FEM outcomes. An additional comparison has been made with the exact solution, proposed by Timoshenko, for the case of simply supported beam subjected to a moving force. The comparisons show a good agreement.

Keywords: engineering structures; dynamic response; forced vibrations; bridge crane

1. Introduction

The presence of movable load causes the occurrence of vibrations in civil engineering structures such as bridges, railway lines, bridge cranes and others. The computational scheme of these objects most commonly leads to a beam model under moving load. The occurrence of this engineering problem is connected with the construction and exploitation of railroad installations. Three studies mark the beginning of a solution to this problem. The first mathematical model of the elastic curve of Bernoulli-Euler beam, subjected to a load, moving with a constant horizontal

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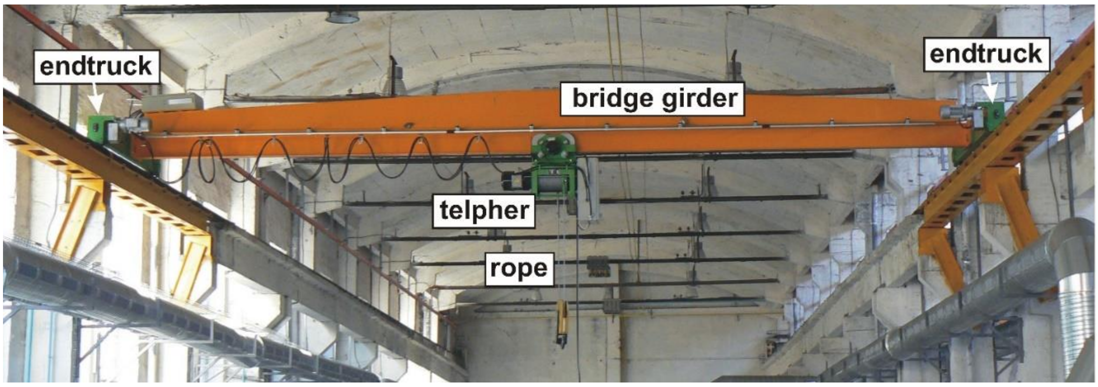

Fig. 1 Single girder bridge crane

therefore, the general solution in infinite sums can be obtained (Timoshenko 1972).

In the engineering practice the technical solutions for fixture of the ends of a beam, subjected to bending, lead to a model with elastic angular supports. A such model of Bernoulli-Euler beam under moving load was developed by Maximov (2014). The elastic angle supports restrict the rotations of the end cross-sections for beam bending, depending on the stiffness of the supports. For instance, in many constructional solutions the bridge girder (principal beam) of a bridge crane is connected in both its ends for the vertical internal faces of the endtrucks (runway beams) through plates and coupling flanges with fitted bolt connections (Fig. 1). The elasticity of the angular supports in a vertical plane is a function of the endtrucks torsion stiffness. In terms only of bending, the bridge girder is double statically indeterminated: hyperstatic quantities are the elastic moments in the two additional angular supports.

The moving on the bridge girder telpher with the elastically suspended load is the moving load. This system (bridge girder-telpher-load) departs from the scope of the known modeled tasks, as it assumes two generalized coordinates: deflection of the elastic line of the bridge girder (depending on the time and the abscissa defining a particular cross-section) and the elastic elongation of the “telpher-load” system. In the known methods for calculating the bridge girder of the bridge crane, the dynamic effect of movement on the bridge girder of the “hoist-load” system has been taken into account with a coefficient of dynamism, which is an empirical function of the telpher nominal velocity (Kolarov *et al.* 1986). This dynamic coefficient takes into account all unevenness (which, of course, have a stochastic nature) of the telpher route, which in turn are the reason for dynamic loading. In fact, the mobile “telpher-elastic suspended load” system causes forced vibrations of the bridge girder, which are superimposed on the free vibrations resulting from the random effects of the route unevenness. While the second type of vibrations rapidly subsides due to material hysteresis mostly, the forced vibrations exist during the whole telpher movement.

The main objective of this study is to evaluate those forced vibrations of the “bridge girder-telpher-load” system due to telpher movement along the bridge girder, respectively, to assess the dynamic effect on the bridge girder.

In this study, the bridge girder is modeled as Bernoulli-Euler beam with angular elastic supports and with constant cross-section, respectively, with constant mass per unit length. The “telpher-load” system is moved with constant horizontal velocity along the bridge girder. The connection between the telpher and the load is linear elastic. All masses (bridge girder, telpher, load) are taken into account.

is the telpher mass; H is heaviside function; ς is damping coefficient of distributed linear damping; N^r is the elastic force in the rope; M^{ℓ} is the load mass; g is gravity acceleration; $z(t)$ is the rope elongation; v is the telpher horizontal velocity; c^r is the rope stiffness ($\ell_r^{in} = const$); ℓ_r^{in} is the initial rope length.

The initial and boundary conditions are

$$w(x,0) = w_0(x);\ \eta(x,0) = w_0(x) + \frac{M^{\ell} g}{c^r} \tag{2}$$

$$\begin{aligned} &w(0) = w(\ell) = 0; \\ &EJ\frac{\partial^2 w(0)}{\partial x^2} - c_{\varphi}\frac{\partial w(0)}{\partial x} = 0; \\ &EJ\frac{\partial^2 w(\ell)}{\partial x^2} + c_{\varphi}\frac{\partial w(\ell)}{\partial x} = 0 \end{aligned} \tag{3}$$

The system (1) describing the "bridge girder-telpher-load" system dynamic response, cannot be integrated analytically. Two approaches are possible: numerical solution; development and implementation of appropriate engineering approach. The numerical solution requires a particular geometry and configuration of the mechanical system and does not always allow a thorough analysis to be conducted. In this study, the second approach has been adopted.

3. Essence of the proposed approach

In this section, an engineering approach for modeling the elastic line of Bernoulli-Euler beam with elastic angular supports is proposed. The method is based on the application of the infinite trigonometric series. A straight beam with elastic angular supports with stiffness equal to c_{φ}, limiting the bending rotations of the end cross-sections (Fig. 2), is considered. The beam elastic line lies in the xw plane. The $w(t, x)$ dynamic deflection is presented as

$$w(t,x) = \varphi(t) y(x) \tag{4}$$

where $\varphi(t)$ is normal coordinate and $y(x)$ is normal function.

In order to define the $y(x)$ normal function, the $w(t, x)$ dynamic deflection is considered in a static regime, i.e.,: $w=w(x)$.

The deflection $w(x)$ must satisfy the condition $w(0) = w(\ell) = 0$, but $w'(0) \neq 0$, $w'(\ell) \neq 0$, $w''(0) \neq 0$, $w''(\ell) \neq 0$, as between $w'(0)$ and $w''(0)$, respectively between $w'(\ell)$ and $w''(\ell)$, a correlation exists: for a particular angle of rotation of the end cross-section, there is a specific elastic moment. The expression for deflection $w(x)$ of the beam elastic line is offered in the form

$$\begin{aligned} w(x) = &\sum_{n=1}^{n=\infty} A_n\left(1 - \cos\frac{2n\pi x}{\ell}\right) + \\ &+ \sum_{n=1,3,5,\ldots}^{n=\infty} B_n \sin\frac{n\pi x}{\ell} \end{aligned} \tag{5}$$

$$w(x) = \sum_{n=1}^{n=\infty} A_n \left(1 - \cos\frac{2n\pi x}{\ell}\right) + \\ + \frac{k\pi}{4} \sum_{n=1,3,5,...}^{n=\infty} nA_n \sin\frac{n\pi x}{\ell} \tag{12}$$

The unknown coefficients A_n can be determined, for example through the principle of possible displacements for equilibrium position depending on the particular load.

The normal $\varphi(t)$ coordinate in Eq. (4) is actually the A_n coefficient in Eq. (12), when n=1 and the deflection depends on the time, i.e.,: $w=w(x, t)$. Two arguments exist in favor of this assumption (n=1):

• In the engineering structures the elastic curve of a two-supported beam usually corresponds to its basic eigentone under free vibrations;

• The practice shows that the bending stresses in a two-supported beam are biggest when the load is equally distant from both supports.

The normal $y(x)$ function is obtained from (12) after substitution of A_n=1 and n=1

$$y(x) = \left(1 - \cos\frac{2\pi x}{\ell}\right) + \frac{k\pi}{4}\sin\frac{\pi x}{\ell} \tag{13}$$

When k=0 (beam with both ends fixed), the second addend in (13) is removed: $y(x) = \left(1 - \cos\frac{2\pi x}{l}\right)$. When $k=\infty$ (simply supported beam), the normal function is $y(x) = \sin\frac{\pi x}{l}$. Any further transformations were made with the assumption that k is a finite number greater than zero.

4. Forced vibrations of the "bridge girder-telpher- load" system

The developed approach has been used in order to study the dynamic response of the "bridge girder-telpher-load" system due to telpher motion. The equations of motions of this system can be presented as

$$\begin{aligned} &\frac{d}{dt}\frac{\partial E_k}{\partial \dot{\varphi}} - \frac{\partial}{\partial \varphi}\left(E_k - E_p\right) = Q_\varphi \\ &\frac{d}{dt}\frac{\partial E_k}{\partial \dot{z}} - \frac{\partial}{\partial z}\left(E_k - E_p\right) = Q_z \\ &\xi = vt \end{aligned} \tag{14}$$

where

$$E_k = E_k^b + E_k^h + E_k^\ell \tag{15}$$

Thus, the nonlinear Eq. (14) are approximated by a system of linear differential equations of the second order, which has an analytical solution. Simultaneously a certain error is introduced: when the telpher is in the beam middle, the computed natural frequencies are higher than the actual ones and vice versa: the computed frequencies will be smaller when the telpher is positioned at the beam ends.

The system potential energy is

$$E_p = E_p^b + E_p^{elas} + E_p^r \tag{24}$$

where E_p^b, E_p^{elas} and E_p^r are components respectively from the bridge girder, the elastic angular supports and the elastic rope.

The sum of the first two components is

$$\begin{aligned} E_p^b + E_p^{elas} &= \frac{EJ}{2}\varphi^2 \int_0^{\ell} [y''(x)]^2 dx + c_\varphi [y'(0)]^2 \varphi^2 = \\ &= \frac{EJ\pi^4}{l^3}\left(4 + \frac{5}{3}k + \frac{\pi^2 k^2}{64}\right)\varphi^2 \end{aligned} \tag{25}$$

The elastic rope potential energy is

$$E_p^r = \frac{1}{2} c^r z^2 \tag{26}$$

The external potential forces of the "bridge girder-telpher-load" system are: Q^ℓ - the load weight; $q=\rho Fg$ - the distributed load of the beam weight, where $\vec{g}$ - the gravitational acceleration; G^h - the telpher weight. An increase of the w deflection, equal to δw, is assigned which leads to a distortion of the beam elastic line (the z coordinate remains constant)

$$\delta w = \delta\varphi . y(x) \tag{27}$$

The applied points of the forces G^h and Q^ℓ perform displacements, respectively

$$\delta w_{G^h} = \delta w_{Q^\ell} = \delta\varphi . F(t) \tag{28}$$

The virtual works of the G^h and Q^ℓ forces, and the distributed load q, are respectively

$$\delta A\left(G^h + Q^\ell\right) = \left(G^h + Q^\ell\right) F(t) \delta\varphi \tag{29}$$

$$\delta A(q) = \int_0^{\ell} q \partial w dx = q\ell\left(1 + \frac{k}{2}\right)\delta\varphi \tag{30}$$

The virtual work of the generalized force Q_φ is

$$\delta A(Q_\varphi) = Q_\varphi . \delta\varphi \tag{31}$$

From the virtual work of the forces G^h and Q^ℓ, and the distributed load q

where $d = \dfrac{M^l}{c^r}$ is the reciprocal value of the square of the natural frequency of the "rope-load" system.

After summing the two Eqs. (37)

$$\left(\frac{\overline{M}_\varphi}{\overline{M}_z}-1\right)\ddot{\varphi}+\left(1-\frac{\overline{M}_z}{\overline{M}_\varphi}\right)\ddot{z}+A_\varphi\left(\frac{1}{\overline{M}_z}-\frac{1}{\overline{M}_\varphi}\right)\varphi = \\ =\left[F(t)Q_\Sigma+G_\Sigma\right]\left(\frac{1}{\overline{M}_z}-\frac{1}{\overline{M}_\varphi}\right) \tag{38}$$

Both sides of the second of the Eq. (37) is differentiated twice and the obtained equation is solved with respect to $\ddot{z}$

$$\ddot{z}=d\left(\frac{\overline{M}_\varphi}{\overline{M}_z}-1\right)\frac{d^4\varphi}{dt^4}+d\,\frac{A_\varphi}{\overline{M}_z}\ddot{\varphi}-\ddot{F}(t).\frac{Q_\Sigma}{\overline{M}_z}.d \tag{39}$$

After substitution of (39) in (38), the differential equation of the $\varphi(t)$ normal coordinate is obtained

$$\frac{d^4\varphi}{dt^4}+2ab\ddot{\varphi}+ac\varphi=F(t)aQ_\Sigma+aG_\Sigma+adQ_\Sigma\ddot{F}(t) \tag{40}$$

where

$$a=\frac{1}{\left(\overline{M}_\varphi-\overline{M}_z\right)d}$$

$$2b=\overline{M}_\varphi+A_\varphi d$$

$$c=A_\varphi$$

The roots of the characteristic equation are

$$r_{1,2}=\pm i\sqrt{ab-\sqrt{a^2b^2-ac}}$$

$$r_{3,4}=\pm i\sqrt{ab+\sqrt{a^2b^2-ac}}$$

For physically acceptable values of the parameters of the "bridge girder-telpher-load" mechanical system the condition

$$ab^2-c>0$$

should always be fulfilled.

$$ab-\sqrt{a^2b^2-ac}=\omega_1^2; \\ ab+\sqrt{a^2b^2-ac}=\omega_2^2 \tag{41}$$

$$\bar{c} = \frac{A_\varphi}{\overline{M}_z}$$

It is assumed that in the t=0 moment the telpher is on the left support. Therefore, the initial conditions are

$$\varphi(0) = 0;\ \dot{\varphi}(0) = 0;$$
$$z(0) = \frac{Q^\ell}{c^r} = \frac{g}{\omega_l^2};\ \dot{z}(0) = 0 \tag{46}$$

From (42) and (44)-(46)

$$C_1 = \frac{-A_1\overline{m}\,\omega_2^2 - A_2\overline{m}\left(\omega_2^2 - 4\Omega^2\right) + \dfrac{G_\Sigma - \overline{M}_z g}{\overline{M}_z} + \dfrac{\omega_1^2 Q^l}{c^r}}{\overline{m}\left(\omega_2^2 - \omega_1^2\right)} \tag{47}$$

$$C_2 = \frac{-A_3\overline{m}\,\Omega\left(\omega_2^2 - 4\Omega^2\right) + \dfrac{k\pi\Omega Q_\Sigma}{4\overline{M}_z}}{\overline{m}\left(\omega_2^2 - \omega_1^2\right)\omega_2} \tag{48}$$

$$C_3 = \frac{A_1\overline{m}\,\omega_1^2 - A_2\overline{m}\left(\omega_1^2 - 4\Omega^2\right) + \dfrac{G_\Sigma - \overline{M}_z g}{\overline{M}_z} + \dfrac{\omega_1^2 Q^l}{c^r}}{\overline{m}\left(\omega_2^2 - \omega_1^2\right)} \tag{49}$$

$$C_4 = \frac{A_3\overline{m}\,\Omega\left(\omega_1^2 - 4\Omega^2\right) - \dfrac{k\pi\Omega Q_\Sigma}{4\overline{M}_z}}{\overline{m}\left(\omega_2^2 - \omega_1^2\right)\omega_2} \tag{50}$$

follows for the C_i integration constants.

The constants C_1 and C_3 are presented as

$$C_i = C_i^{free} + C_i^{forced},\ i = 1{,}3 \tag{51}$$

where

$$C_i^{free} = \frac{\omega_1^2 Q^l}{\overline{m}\left(\omega_2^2 - \omega_1^2\right)c^r} = \frac{g}{\overline{m}\left(\omega_2^2 - \omega_1^2\right)},\ i = 1, 3 \tag{52}$$

$$C_1^{forced} = \frac{-A_1\overline{m}\,\omega_2^2 - A_2\overline{m}\left(\omega_2^2 - 4\Omega^2\right) + \dfrac{G_\Sigma - \overline{M}_z g}{\overline{M}_z}}{\overline{m}\left(\omega_2^2 - \omega_1^2\right)} \tag{53}$$

$$C_3^{forced} = \frac{A_1\overline{m}\,\omega_1^2 + A_2\overline{m}\left(\omega_1^2 - 4\Omega^2\right) - \dfrac{G_\Sigma - \overline{M}_z g}{\overline{M}_z}}{\overline{m}\left(\omega_2^2 - \omega_1^2\right)} \tag{54}$$

The numerical results are shown for a single girder bridge crane with bridge girder length $\ell = 20m$, maximum lifting capacity *Q*=50 *kN* and coefficient *k*=11.33 (see Eq. (11)), respectively stiffness of the angular supports c_φ=24489156.8 *Nm*/*rad*. The sizes of the bridge girder cross-section (Fig. 3) are: *u*=0.015 *m*; *v*=0.02 *m*; *a*=0.06 *m*; *b*=0.395 *m*; *h*=0.595 *m*; *F*=0.0324 m^2; J_y=0.001734 m^4. The telpher mass is m^h=1000 *kg*. The bridge girder density and the rope stiffness are respectively ρ=7850 kg/m^3 and c^r=107 *N*/*m*.

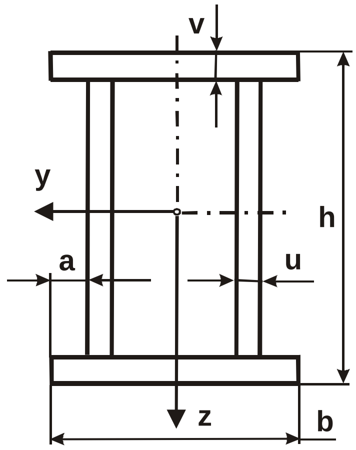

Fig. 3 Cross-section of the bridge girder

Fig. 4 visualizes the components of the function of forced vibrations (55) caused by the horizontal telpher velocity *v*=0.5 *m*/*s* as well as free vibrations and static deflection of the bridge girder middle cross-section. The time interval corresponds to the telpher location in vicinity of the middle section: in the interval ±2 *m* from the middle cross-section. Apparently, the contribution of the forced vibrations with own frequencies for obtaining the maximum dynamic deflection is greatest (Fig. 4).

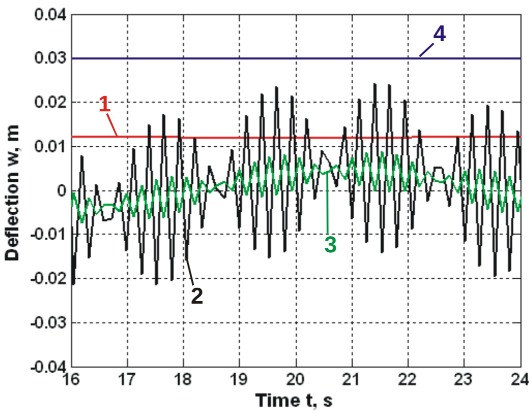

Fig. 4 Forced vibrations, free vibration and static deflection: 1 - forced vibrations with forced frequency Ω; 2 - forced vibrations with natural frequencies ω_i; 3 - free vibrations; 4 - static deflection

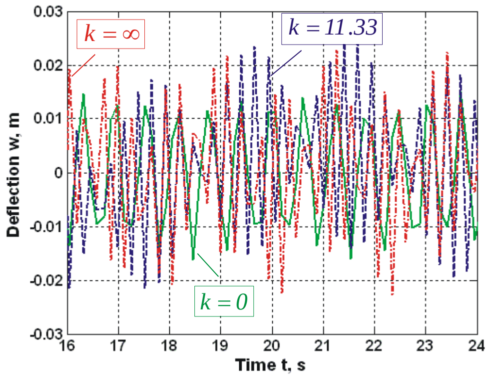

Fig. 7 Effect of the coefficient k on the forced vibrations with natural frequencies ω_i

Fig. 8 shows the effect of the coefficient k on the dynamic coefficient k_d, defined for the beam middle cross-section. The coefficient k_d is largest for fixed beam, k=0 (Fig. 8(c)), and significantly exceeds the dynamic coefficient k_d=1.04+0.06v due to the telpher motion, shown in (Kolarov *et al.* 1986). With the increase of k, the dynamic coefficient k_d sharply decreases. Of course, the cases where k=∞ (Fig. 8(a)) and k=0 (Fig. 8(c)) have a theoretical significance only. For practical real values of k (see Fig. 8(b)), the dynamic coefficient k_d calculated in

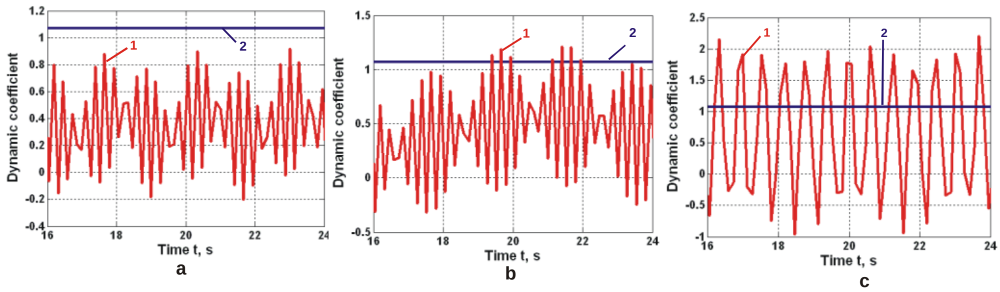

Fig. 8 Effect of the coefficient k on the dynamic coefficient k_d: 1 - dynamic coefficient according to (58); 2 - dynamic coefficient according to [47]; (a) k=∞; (b) k=11.33; (c) k=0

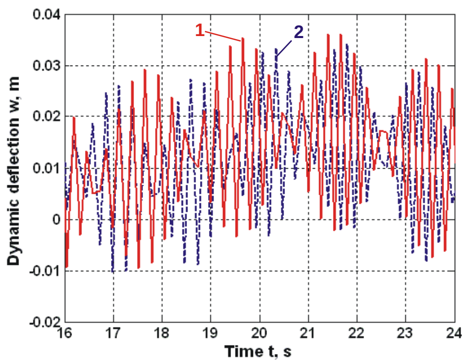

Fig. 9 Comparison of the forced vibrations for the middle beam section: (a) $F(t)=\overline{F}$; (b) for the actual value of the $F(t)$ function

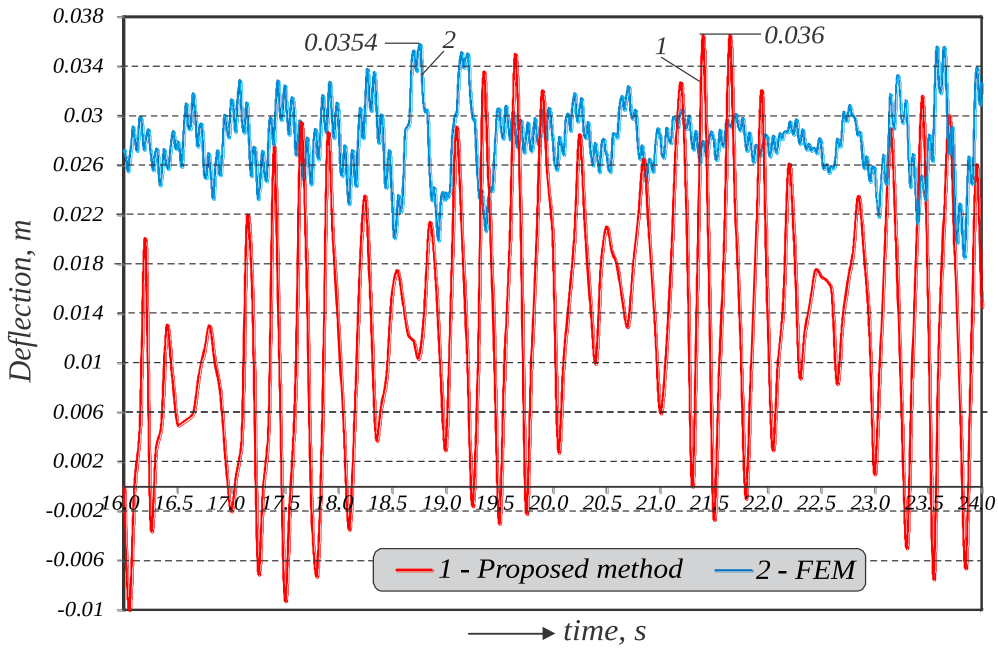

Fig. 11 Deflection of the middle section centre: 1. proposed method; 2. FEM

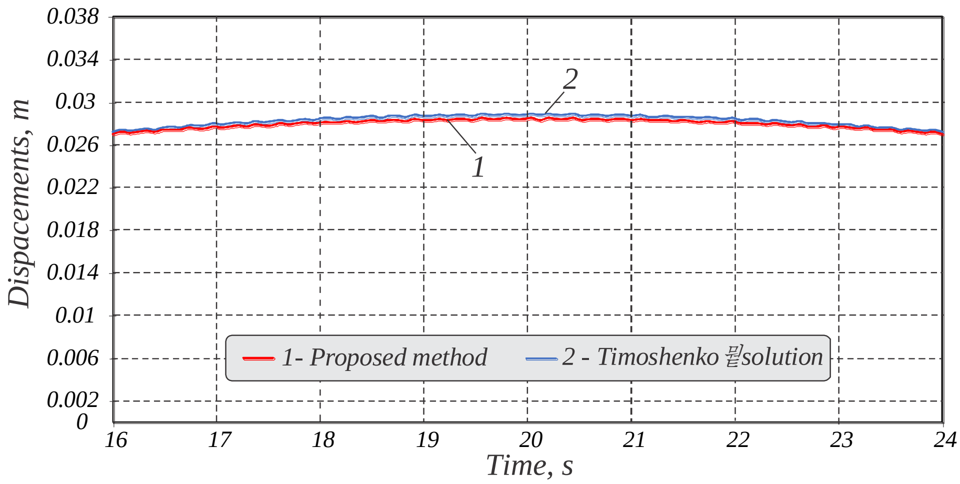

Fig. 12 Deflection of the middle section centre of simply supported beam: 1. proposed method; 2. Timoshenko

FEM outcomes for the displacement of the middle cross-section centre (point *A*) of the beam are depicted in Fig. 11. The comparison of the obtained FEM results with those from the proposed method shows good agreement with respect to the maximum deflection: 0.0354 m versus 0.0360 m.

An additional comparison has been made with the exact solution, proposed by Timoshenko, for the case of simply supported beam subjected to a moving constant force. Tymoshenko's solution in an infinite series is (Timoshenko 1972)

• The rope 3 is assumed to be rigid (non-deformable);
• The moving constant force P has a magnitude: $P = G^h + Q^\ell$.

Using the numeric data from Section 6, the graphs of Eqs. (59) and (60) for $x = \frac{\ell}{2}$ (middle cross-section) are shown in Fig. 12. The comparison shows a very good agreement between the solutions.

8. Conclusions

• A method for separation of the variables (time, abscissa) in the differential equation of the elastic line of Bernoulli-Euler beam has been developed. Unlike Fourier's method, the proposed method divides the variables before drawing up the differential equation. For this purpose, the elastic line of the beam is modeled in advance depending on the boundary conditions through the infinite trigonometric series method. The developed method can be utilized in many engineering applications, leading to "a beam under moving load model".

• The forced vibrations of the "bridge girder-telpher-load" system of single girder bridge crane, due to telpher motion along bridge girder, have been established and analyzed through the developed method. A conclusion has been made that the normal stress at a critical point form the bridge girder middle cross-section will change at an asymmetric cycle, similar to the dynamic deflection.

• The concept of "dynamic coefficient" has been introduced, which is a ratio of the dynamic deflection of the principal beam, due to the forced vibrations, to the static one. This ratio has been compared with the known from literature empirical dynamic coefficient during the telpher movement, due to the track unevenness. The dynamic coefficient k_d, calculated in accordance with Eq. (58), shows larger values than that in (Kolarov *et al.* 1986), and it actually reveals the mechanism of the bridge girder loading: the loading resulting from the telpher uniform movement along the bridge girder is cyclical. Therefore, the telpher movement along the bridge girder is a prerequisite for nucleation and propagation of fatigue cracks. The introduced dynamic coefficient has to be taken into account for engineering calculations of the bridge crane metal structure.

• In order to verify the degree of approximation, the obtained results have been compared with FEM outcomes. An additional comparison has been made with the exact solution, proposed by Timoshenko, for the case of simply supported beam subjected to a moving force. The comparisons show a good agreement.

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References

Amiri, S.N. and Onyango, M. (2010), "Simply supported beam response on elastic foundation carrying repeated rolling concentrated loads", *J. Eng. Sci. Technol.*, **5**(1), 52-66.

load with different boundary conditions", *J. Sharif Univ. Technol. Trans. B: Mech. Eng.*, **16**(3), 273-279.
Michaltos, G.T., Sophianopulos, D. and Kounadis, A.N. (1996), "The effect of a moving mass and other parameters on the dynamic response of a simply supported beam", *J. Sound Vibr.*, **191**(3), 357-362.
Michaltos, G.T. (2002), "Dynamic behaviour of a single-span beam subjected to loads moving with variable speeds", *J. Sound Vibr.*, **258**(2), 359-372.
Nikkhoo, A. and Amankhani, M. (2012), "Dynamic behaviour of functionally graded beams transferred by a moving random load", *Ind. J. Sci. Technol.*, **5**(12), 3727-3731.
Omolofe, B. (2013), "Deflection profile analysis of beams on two-parameter elastic subgrade", *Lat. Am. J. Sol. Struct.*, **10**(2), 263-282.
Piccardo, G. and Tubino, F. (2012), "Dynamic response of Euler-Bernoulli beams to resonant harmonic moving loads", *Struct. Eng. Mech.*, **44**(5), 681-704.
Prager, W. and Save, M. (1963), "Minimum-weight design of beams subjected to fixed and moving loads", *J. Mech. Phys. Sol.*, **11**(4), 255-267.
Rostam, M.R., Javid, F., Esmailzadeh, E. and Younesian, D. (2015), "Vibration suppression of curved beams traversed by off-center moving loads", *J. Sound Vibr.*, **352**, 1-15.
Samani, F. and Pellicano, F. (2009), "Vibration reduction on beams subjected to moving loads using linear and nonlinear dynamic absorbers", *J. Sound Vibr.*, **325**(4-5), 742-754.
Soares, R.M., Del Prado, Z.J.G.N. and Goncales, P.B. (2010), "On the vibration of beams using a moving absorber and subjected to moving loads", *Mecan. Comput.*, **29**, 1829-1840.
Sun, L. and Luo, F. (2008), "Study-state dynamic response of a Bernoulli-Euler beam on a viscoelastic foundation subjected to a platoon of moving dynamic load", *J. Vibr. Acoust.*, **130**, 051002-1-051002-19.
Thambiratnam, D. and Zhuge, Y. (1996), "Dynamic analysis of beams on an elastic foundation subjected to moving loads", *J. Sound Vibr.*, **198**(2), 149-169.
Timoshenko, S.P. (1922), "On the forced vibrations of bridges", *Philosoph. Mag. Ser. 6*, **43**(257), 1018-1019.
Timoshenko, S.P. (1972), *Theory of Elasticity*, Naukova Dumka, Kiev, Russia.
Willis, R. (1849), *Report of the Commissioners Appointed to Inquire into the Application of Iron to Railway Structures*, William Clowes & Sons, London, U.K.
Wu, J.J. (2005), "Dynamic analysis of inclined beam due to moving load", *J. Sound Vibr.*, **288**(1-2), 107-131.
Xia, H., Zhang, N. and Guo, W.W. (2006), "Analysis of resonance mechanism and conditions of train-bridge system", *J. Sound Vibr.*, **297**(3-5), 810-822.
Yang, Y.B., Yau, J.D. and Hsu, L.C. (1997), "Vibration of simples beams due to trains moving at high speeds", *Eng. Struct.*, **19**(11), 936-944.
Yau, J.D. (2004), "Vibration of simply supported compound beams to moving loads", *J. Mar. Sci. Technol.*, **12**(4), 319-328.
Zehsaz, M., Sadeghi, M.H. and Asl, A.Z. (2009), "Dynamic response of railway under a moving load", *J. Appl. Sci.*, **9**(8), 1474-1481.
Zheng, D.Y., Cheung, Y.K., Au, F.T.K. and Cheng, Y.S. (1998), "Vibration of multi-span non-uniform beams under moving loads by using modified beam vibration functions", *J. Sound Vibr.*, **212**(3), 455-467.
Zibdeh, H.S. and Rackwitz, R. (1995), "Response moments of an elastic beam subjected to Poissonian moving loads", *J. Sound Vibr.*, **188**(4), 479-495.

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