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Effect of strain ratio variation on equivalent stress block parameters for normal weight high strength concrete

Prabhat Kumar†

Structural Engineering Division, Central Building Research Institute, Roorkee, 247667 UA, India

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Abstract. Replacement of actual stress distribution in a reinforced concrete (RC) flexural member with a simpler geometrical shape, which maintains magnitude and location of the resultant compressive force, is an acceptable conceptual trick. This concept was originally perfected for normal strength concrete. In recent years, high strength concrete (HSC) has been introduced and widely used in modern construction. The stress block parameters require updating to account for special features of HSC in the design of flexural members. In future, more varieties of concrete may be developed and a corresponding design procedure of RC flexural members will be required. The usual practice is to conduct large number of experiments on various sizes of specimen and then evolve an empirical relation. This paper presents a numerical procedure through which the stress block parameters can be numerically derived for a given strain ratio variation. The material model for concrete is presented and computational procedure is described. This procedure is illustrated with several variations of strain ratio. The advantages of numerical procedure are that it costs less and it can be used with new material models for any new variety of concrete.

Keywords: beam; bending; computer methods; flexure strength; high strength concrete; reinforced concrete; safety factor; stress block; structural analysis; ultimate strength.

1. Introduction

Concrete is a widely used construction material. The conventional ultimate strength calculations of flexural members with non-rectangular cross section based on basic principles of structural mechanics becomes cumbersome and lengthy. This problem is further compounded when high strength concrete (HSC) is involved because its analytical representation in itself is a very complex problem. These problems mask the physical basis of design and result in lack of understanding. The equivalent rectangular stress block (ERSB) formulation is a conceptual trick through which the general analysis is easily applied to cases of greater complexity. This formulation creates greater visibility without violating fundamental principles of structural mechanics.

The actual geometrical shape of compression stress distribution may be complex as well as variable. However, its complete and precise knowledge is not required if the magnitude of the compression force F_c and its location are known. Now any other convenient geometrical shape may replace the actual distribution. If the new geometrical shape maintains magnitude and location of F_c, the final answer is not affected.

† Scientist, E-mail: anilsw@sancharnet.in

The ERSB concept originated in USA and it was meant for flexural members of normal strength concrete. It is covered in ACI code of practice (ACI 2004). It is adopted throughout the world in one form or another. Usually ERSB parameters are experimentally derived, which is expensive and time consuming. This paper presents an analytical approach as against the existing empirical method. The ERSB parameters are derived for several different distributions of ultimate to peak concrete strain ratio, which is termed in this paper as strain ratio. Although, this paper deals with HSC, the analytical procedure applies equally conveniently to any new material, the stress-strain relation of which can be numerically represented. The numerical results derived in this paper are in excellent agreement with published information on the subject.

2. Research significance

The concept of Equivalent Rectangular Stress Block (ERSB) is widely used in the design and analysis of reinforced concrete members. After introduction of high strength concrete (HSC), several versions of empirical ERSB provisions are in current use. This ambiguity defeats the basic purpose of ERSB formulation. A numerical procedure, which applies over a wide range of concrete strength, is used in this paper to derive ERSB parameters. The results of computation are highly sensible. A comparison of empirical and numerical values of ERSB parameters may eliminate ambiguity, uncertainty and diversity of opinion. It may also re-establish uniqueness and simplicity of ERSB formulation, which was the basic idea behind its formulation. This numerical procedure costs less and it can always be updated with new material model for concrete. This paper shows that specification of ERSB parameters implicitly assumes variation of ultimate to peak strain ratio. But the empirical procedures fail to describe it and it will never be known. The numerical procedure presented in this paper eliminates this gap. Further, ERSB parameters define balanced reinforcement ratio, which separates under and over reinforced beams. Therefore, a proper definition of ERSB parameters is necessary for a safe design of flexural members.

3. State-of-the-art

3.1. Material models of concrete

The parabolic model (Hognestad 1951), exponential model (Smith and Young 1955) and Desai and Krishnan model (Desai and Krishnan 1964) are applicable to concrete below 40 MPa strength. Relatively more recent material models (Attard and Setunge 1996, Wee, *et al.* 1996) contain several empirical constants and are not convenient for use in analytical studies. Another more recent material model (Kumar 2003) is free of such defects but it requires solution of a simple nonlinear equation. This solution can be quickly derived on a simple pocket calculator.

3.2. ERSB parameters

The ERSB parameters have been experimentally determined. It requires test specimen of a particular shape and a special test procedure. This subject has been discussed in several recent publications (Hognestad, *et al.* 1955, Mattock, *et al.* 1961, Karr, *et al.* 1978, Kahn and Meyer 1995,

Table 1 Code provisions of ERSB parameters

Source	α	β
ACI 318-2004	0.85	$0.85 - 0.008(f_c' - 30)$ $0.85 \geq \beta \geq 0.65$
Ibrahim and MacGregor (1997)	$0.85 - 0.00125\ f_c'$ $\alpha \geq 0.725$	$0.95 - 0.0025\ f_c'$ $\beta \geq 0.70$
Ozbakkaloglu and Saatcioglu (2004)	$0.85 - 0.0014\ (f_c' - 30) \geq 0.72$	$0.85 - 0.002\ (f_c' - 30) \geq 0.67$
AS 3600-1994	0.85	$0.85 - 0.007(f_c' - 28)$ $0.85 \geq \beta \geq 0.65$
CAN3-M, 1994	$0.85 - 0.0015\ f_c'$ $\alpha \geq 0.67$	$0.97 - 0.0025\ f_c'$ $\beta \geq 0.67$
NZ 3101-1995	$1.07 - 0.004\ f_c'$ $0.85 \geq \alpha \geq 0.75$	$1.09 - 0.008\ f_c'$ $0.85 \geq \beta \geq 0.65$

Ibrahim and MacGregor 1997, Mansur, *et al.* 1997, Attard and Steward 1998, Ozbakkaloglu 2004). The major code provisions on ERSB are summarized in Table 1. It is obvious that the original ERSB concept was introduced for normal strength concrete (NSC) and it was unambiguous. The diversification noticed in Table 1 is mainly due to adoption of HSC.

3.3. Theory

Fig. 1 introduces the basic ERSB concept for a rectangular beam and most of the notation used in this paper.

3.4. Material model

The following concrete material model is used in this paper (Kumar 2003). The stress f and strain ε are used in non-dimensional form as $Y = f/f_P$ and $X = \varepsilon/\varepsilon_P$ where subscript 'P' denotes peak values.

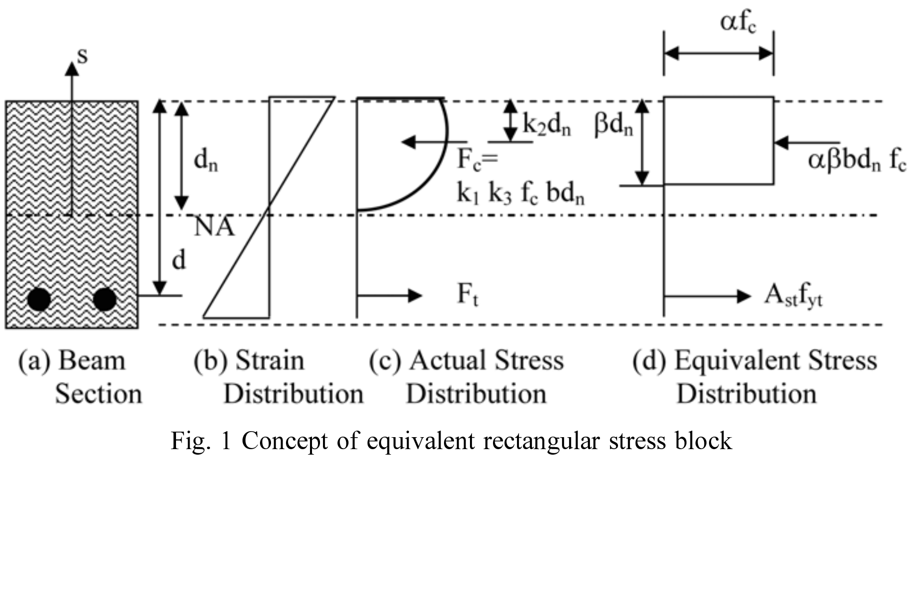

Fig. 1 Concept of equivalent rectangular stress block

$$Y = \frac{AX}{B + CX + DX^n} \tag{1}$$

Starting with Eq. (1) and imposing boundary conditions at origin (X=0, Y=0 and $m = dY/dX = E_o/E_P$), at peak point (X=1, Y=1 and dY/dX=0) and at inflexion point ($X=X_I$ $Y=Y_I$ and d^2Y/dX^2=0), Eq. (2) is obtained.

$$Y = \frac{m(n-1)X}{n-1+[m(n-1)-n]X+X^n} \tag{2}$$

The subscripts o and I denotes origin and inflexion point, respectively. Where m is the ratio of initial tangent modulus to peak secant modulus, which enforces slope compatibility at origin. For condition $[m(n-1) = n]$, Eq. (2) reduces to Eq. (3).

$$Y = \frac{nX}{n-1+X^n} \tag{3}$$

The curvature above and below the inflexion point on the descending branch of the stress strain curve are of opposite sign, therefore, the curvature must vanish at inflexion point. This condition gives Eq. (4), which is used to determine value of parameter n.

$$X_I^n = n+1 \tag{4}$$

Eq. (5) is an approximate solution of Eq. (4) for concrete strength up to 100 MPa.

$$n = 1.7512 \text{ EXP}(0.0286 f_P) \tag{5}$$

The inflexion point may be located from Eqs. (6) and (7) (Attard and Setunge 1996).

$$X_I = 2.50 - 0.30 \, Ln\,(f_P) \tag{6}$$

$$Y_I = 1.41 - 0.17 \, Ln\,(f_P) \tag{7}$$

Wang, *et al.* (1978) give Eq. (8) for stress at inflexion point but corresponding equation for strain is not given.

$$Y_I = 4.0\,(10.0 + f_P)/7.0 \tag{8}$$

Results of Eqs. (7) and (8) are quite comparable. Eq. (6) is used in the present study for solution of Eq. (4).

3.5. ERSB parameters

The equivalence of magnitude and location of the compressive force between actual and its equivalent stress blocks for a general non–rectangular cross section (Fig. 2) are mathematically stated in Eqs. (9) and (10) (Collins and Mitchell 1991).

$$\int_0^{d_n} f(s).b(s).ds = \alpha.f'_c\,A_\beta \tag{9}$$

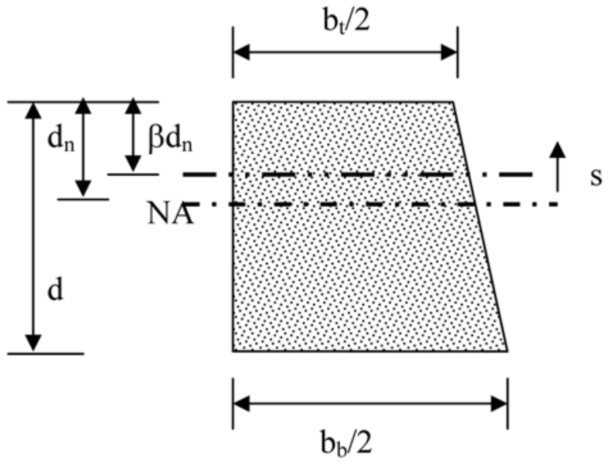

Fig. 2 Shape of beam cross section (half)

$$\bar{y} = \frac{\int_0^{d_n} f(s).b(s).s.ds}{\int_0^{d_n} f(s).b(s).ds} = d_n - d_\beta \tag{10}$$

A rectangular as well as a triangular section can be derived from the section considered in this analysis. The coordinate 's' is measured from neutral axis (NA) into the compression block; A_β and d_β are the area and depth of center of gravity of general beam cross section of βd_n depth (Fig. 2), which are $\beta b d_n$ and $\beta d_n/2$, respectively for a rectangular cross section. Parameter α governs strength mobilized at the ultimate stage and accounts for difference in shape, size and rate of loading in the actual structure and the control specimen. Parameter β determines lever arm to be used in the calculation of flexure strength. The product of these parameters governs magnitude of the compressive force F_c.

3.6. Self-adaptive numerical integration

A self-adaptive Simpson numerical integration scheme is devised in which the number of sub-divisions of the integration interval is progressively doubled till two successive results match within a prescribed tolerance. A part of information required at a step is derived from the already available results of the previous step. This is achieved without proportional increase in the computation effort (Kumar 2004b) because duplication is avoided.

3.7. Depth of neutral axis at ultimate stage

The measurement of depth of neutral axis at ultimate stage is not easy because extensive cracking in the test specimen may influence readings of the strain gages. Sarkar, *et al*. (1997) and Bernardo and Lopes (2004) reported depth of neutral axis at ultimate stage of their test specimen of

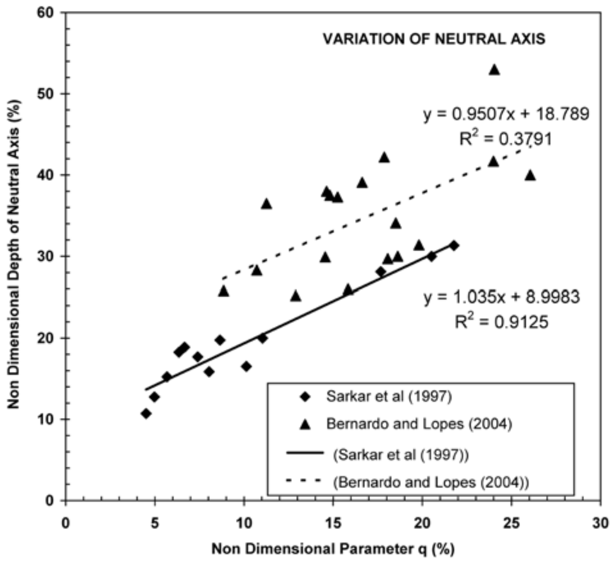

Fig. 3 Variation of neutral axis with geometrical and strength properties of beams

rectangular cross section. Fig. 3 shows variation of depth of neutral axis (ratio of depth of neutral axis with the effective depth) as a function of parameter q ($= A_s\, f_y / bd\, f_c$). A special feature of parameter q is that it includes most of the important factors, which may influence strength of a RC beam. Fig. 3 also shows the equation of linear fit through the data. It is found that for fixed beam and reinforcement properties, depth of neutral axis reduces as concrete strength increases (q decreases). Thus, parameter β decreases and lever arm increases.

3.8. Strain ratio

The ratio of ultimate to peak concrete strain defines strain ratio X_o. Experimental evidence suggests that the peak concrete strain increases, whereas ultimate concrete strain decreases, with increasing concrete strength (Karr, *et al.* 1978, Ibrahim and MacGregor 1997, Mansur, *et al.* 1997). Therefore, the strain ratio should also decrease as concrete strength increases.

Eqs. (9) and (10) can be analytically solved provided the concrete material model is expressed in a form suitable for integration. Previous analysis (Kumar 2004a) showed that dimensions do not govern value of ERSB parameters when beam section is either rectangular or triangular. Even the individual values of ultimate and peak strains are not required. Their ratio X_o is the primary governing factor. For parabolic (Hognestad 1951) material model $Y = f(X) = 2X - X^2$, $\alpha\beta = X_o\,(3-X_o)/3$ and $\beta = (4 - X_o)/(6 - 2X_o)$. It checks with the solution given in Collins and Mitchell (1991). For more complex material models such as the one used in present study (Kumar 2003), numerical integration is required even for beams of rectangular cross section but individual strain values are still not required.

In the present research, value of X_o = 1.75 applies to 20 MPa concrete (ultimate strain 0.0035 and peak strain 0.002) and this value reduces to 1.0 for 140 MPa concrete (ultimate strain practically equal to peak strain). This selection automatically ensures that the shape of actual stress block is triangular for ultra high strength concrete. Fig. 4 gives different strain ratio variations, which are

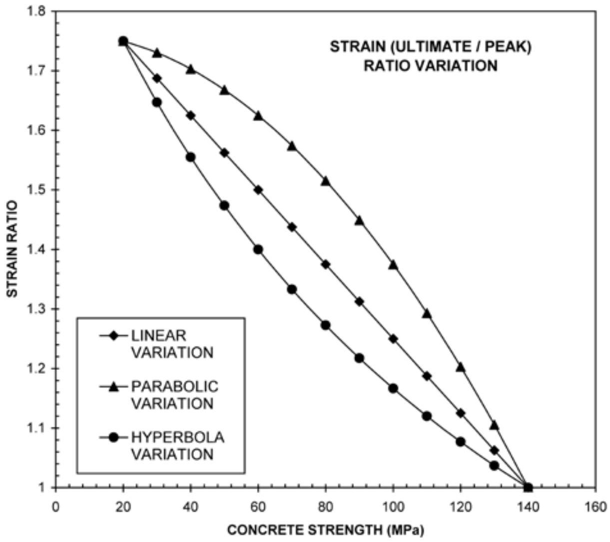

Fig. 4 Variations of strain ratio used in analysis

used in this study. These variations are derived as follows: Linear X_o = (300 – f_P)/160; Parabola X_o = (33900 – 0.75f_P^2)/19200 and Hyperbola X_o = 280/(140 + f_P). Any other variation of strain ratio as a function of concrete strength is equally acceptable. This aspect is under investigation at present and several different opinions exist Eurocode 2 (CEB 1995). Triangular beam section can be similarly analyzed but this paper presents results of only rectangular beam sections.

4. Analysis

The analytically computed ERSB parameters α, β and $\alpha\beta$ for beams of rectangular section are plotted in Figs. 5 to 7. The existing and modified ERSB provisions, along with experimental results

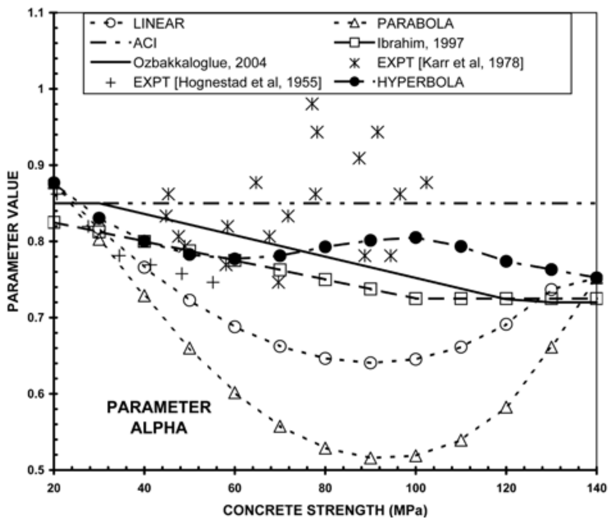

Fig. 5 Analytical and empirical variation of parameter

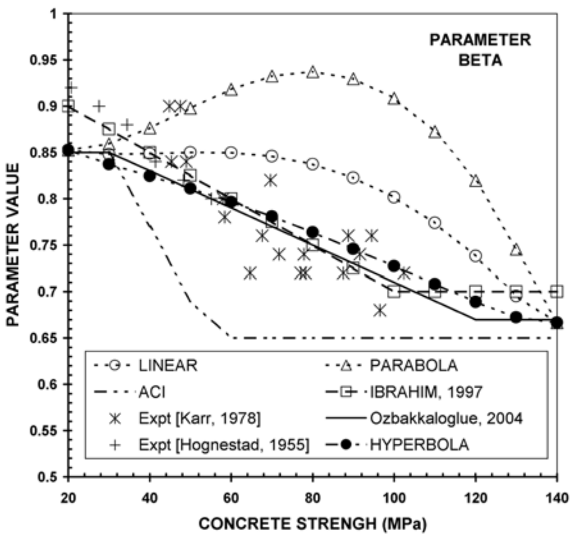

Fig. 6 Analytical and empirical variation of parameter

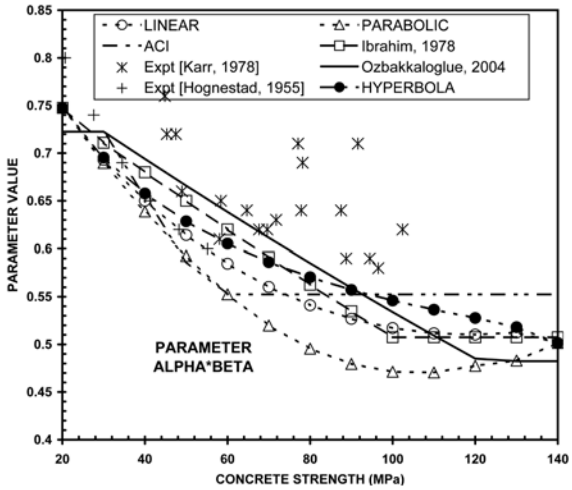

Fig. 7 Analytical and empirical variation of parameter

(Hognestad, *et al*. 1955, Karr, *et al*. 1978), are also shown in Figs. 5 to 7 for the purpose of comparison. The analytical values of parameters α and β are consistent with the existing provisions at the extreme ends, however, the distribution in between depends upon the strain ratio variation. The distribution of parameter $\alpha\beta$ (Fig. 7) agrees with the existing ACI provision in the normal concrete strength range, and with the modified provision (Ibrahim and MacGregor 1997) in the high strength range.

When either the loading is within the elastic range, or concrete strength is very high, the compression stress distribution is very nearly triangular. Theoretical values of ERSB parameters for triangular stress block are $\alpha = 3/4$ and $\beta = 2/3$ (for a rectangular cross section), and $\alpha = 16/27$ and $\beta = 3/4$ (for a triangular cross section). These values are achieved in the analysis for concrete strength of 140 MPa. These features of this analytical study may validate the analysis and its results.

Collins and Mitchell (1991) also solved Eqs. (9) and (10), but did not mention any specific strain ratio variation. The results of parabolic strain ratio variation in Figs. 5 and 6, and results of Collins and Mitchell (1991) are compatible. These clearly are unacceptable as value of parameters α and β deviate from the existing and modified specifications. However, it verifies the analysis presented in this paper.

5. Discussion

It should be noticed in Figs. (5) to (7) that the hyperbolic variation of strain ratio gives the best agreement with the published results on this subject. This fact was not highlighted in proposing modifications based purely on the analysis of experimental data. Another important feature of analytical results is that these are closer to the proposed modifications than the ACI recommended values. These facts must be remembered in future revisions of codes of practice.

The variation of parameter α in Fig. 5 is slightly oscillatory with average value of 0.776-0.8. The analytical values are closer to the proposed modifications than the ACI recommended value. Value of parameter α decreases with increasing concrete strength. From Eq. (9), it is easy to see that value of α depends upon the area under the stress strain curve between origin and the cut off strain. For rectangular section, value of α is proportional to the area under the stress strain curve. Effect of increase in concrete strength is to increase the vertical dimension and to reduce lateral extent of stress strain curve. The net effect is to decrease the area under the curve. So the analytical results and the proposed modifications appear to be reasonable.

The analytical value of parameter β in Fig. 6 is in remarkable agreement with the recently proposed modifications. Published modifications and the results of this study suggest an increase in the value of β over the current ACI recommended value. Selection of appropriate value of parameter β is important in design of flexural members. For heavily reinforced flexural members, the tensile force F_t at ultimate stage of loading is large. The equilibrium of forces requires equally large compressive force F_c. This demands a larger value of parameter β, which cannot be attained due to restrictions of the existing code provisions. It may results in over-estimation of the ultimate strength (Ibrahim and MacGregor 1997). This problem is not likely to arise in lightly reinforced flexural members because appropriate value of parameter β is attainable.

The results of parameter $\alpha\beta$ in Fig. 7 show much less variation than the results in Figs. 5 and 6. Effect of small α and large β does not appear in their product. All results agree well for concrete strength below 55 MPa. The ACI recommended values are smaller in the range of 55 MPa to 100 MPa. Whereas ACI recommended values are higher for ultra HSC.

6. Application example

Consider the beam number 9.0-1.5 from the experimental program of Leslie, *et al.* (1976). Its ultimate moment capacity is 108.8 KN-M. Its calculated strength according to the current ACI provisions is 76.13 KN-M. For beam number 8.0-2 from this experimental program, the ultimate moment capacity is 104.3 KN-M, whereas the calculated moment capacity is 104.33 KN-M. The safety factor (ratio of experimental to calculated value) drops from 1.43 to 1.0 as reinforcement ratio increases from 1.34% to 1.9%. This example highlights deficiency of lever arm parameter β in the current ACI provisions.

7. Conclusions

- The ERSB parameters are analytically derived on the assumption that the strain ratio decreases from 1.75 to 1.0 as concrete strength increases from 20 to 140 MPa. Several variations are considered. The salient features of this study are,
- The numerical procedure of this paper follows fundamental principles of structural mechanics, which are used for the first time to calculate the ERSB parameters.
- The ERSB parameters were originally derived from extensive test results for normal concrete strength. The present analytical procedure establishes its accuracy.
- The hyperbolic variation of strain ratio gives best agreement with the results available in the published literature. Thus, the present study exposes an implicit assumption, which could not have been discovered through the analysis of test results and empirical relationships.
- The variation of parameters α, β, and product $\alpha\beta$ agrees with the current ACI provisions in the

normal concrete strength range, but deviate in the high and ultra high strength range.
- The analytical values of ERSB parameters for 140 MPa concrete agree with the theoretical values for a triangular compression stress block.
- In general, analytical results are closer to the proposed modifications published in recent technical literature than with the ACI recommended values. This verifies the proposed material model as well as the computational strategy proposed in this paper.

Numerical results show that parameters α, β, and product $\alpha\beta$ should decrease with increase in concrete strength. Analysis of experimental data supports this finding. The ACI recommendation on the value of parameter β in the high strength range needs to be re–evaluated.
- The information of this paper provides greater insight in to the ERSB concept. Code writers have to choose a suitable variation in X_o as a function of concrete strength. The procedure of this paper then easily produces ERSB parameters. This numerical procedure applies to any other variety of concrete provided its stress-strain relation is suitable for numerical simulation.

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Notation

A, B, C, D	= Model parameters
A_β	= Area of compression zone
A_{st}	= Area of tension steel
b	= Width of beam
d	= Effective depth of beam
d_n	= Depth of neutral axis (NA)
d_β	= Depth of compression zone centroid from top face
E	= Modulus
F_c	= Compression force
F_t	= Tension force
f	= Stress
f_c	= Concrete strength
f_{st}	= Steel strength
k_1, k_2, k_3	= Real stress block parameters
m	= Ratio of initial tangent to peak secant modulus
n	= Exponent
s	= Coordinate measured from NA
X	= Dimensionless strain
Y	= Dimensionless stress
y	= Depth of compression zone centroid from NA
α, β	= ERSB parameters
ε	= Strain

b	= Bottom face
I	= Inflexion point
P	= Peak point
t	= Top face

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