Computers and Concrete, *Vol. 26, No. 3 (2020) 285-292*
DOI: https://doi.org/10.12989/cac.2020.26.3.285

Analytical solution of free vibration of FG beam utilizing different types of beam theories: A comparative study

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(*Received July 29, 2020, Revised September 2, 2020, Accepted September 11, 2020*)

Abstract. In engineering structures, to having the projected structure to serve all the engineering purposes, the theory to be used during the modeling stage is also of great importance. In the present work, an analytical solution of the free vibration of the beam composed of functionally graded materials (FGMs) is presented utilizing different beam theories. The comparison of supposed beam theory for free vibration of functionally graded (FG) beam is examined. For this aim, Euler-Bernoulli, Rayleigh, Shear, and Timoshenko beam theories are employed. The functionally graded material properties are assumed to vary continuously through the thickness direction of the beam with respect to the volume fraction of constituents. The governing equations of free vibration of FG beams are derived in the frameworks of four beam theories. Resulting equations are solved versus simply supported boundary conditions, analytically. To verify the results, comparisons are carried out with the available results. Parametrical studies are performed for discussing the effects of supposed beam theory, the variation of beam characteristics, and FGM properties on the free vibration of beams. In conclusion, it is found that the interaction between FGM properties and the supposed beam theory is of significance in terms of free vibration of the beams and that different beam theories need to be used depending on the characteristics of the beam in question.

Keywords: FGMs; free vibration; analytical solution; beam theories; comparative study

1. Introduction

Composite materials are formed combining at least two differrent materials, which are insoluble in each other, in macro dimensions to provide new features such as very high strength and stiffness coupled with a very low density, resistance to chemicals, thermal and electrical insulation properties, etc. to the yielding new material suitable for the application areas. Therefore, the theoretical and experimental study of the mechanical behavior of composite structures is one of the interesting topics in the literature. (Patle *et al*. 2018, Mehar *et al*. 2020, Pandey *et al*. 2019, Sahoo *et al*. 2019, Anil *et al*. 2020, Dewangan *et al*. 2020a, b, Sahu *et al*. 2020). However, in composites, the sharp discontinuity between the material properties at the interface of two different types of material may cause vigorous failures in connection with stress concentrations, namely fibers could be separated from the matrix which is called delamination under extreme working conditions (Mahamood *et al*. 2012, Hirwani and Panda 2019).

Therefore, scientists suggested Functionally Graded Materials (FGMs), which eliminate the stress concentration at interfaces with the continuity and gradual variation of material properties (Kouizimi 1993). In FGMs, the volume fraction of two materials changes (generally ceramic and metal) as a function of position throughout a specific dimension of the structure in order to achieve the desired function. In the said structural element, the graded structure of the material keeps metals against corrosion, oxidation, and wear while diminishes imperfections such as interface and surface cracks and failure of the ceramic. FGMs have wide application field in aerospace, automotive, aviation, civil and mechanical engineering structures (Koizumi 1997, Suresh and Mortensen 1998, Kieback *et al*. 2003, Birman and Byrd 2007, Chackraverty and Pradhan 2016, Alsaid-Alwan 2017, Balubaid *et al*. 2019, Kaddari *et al*. 2020, Nejadi and Mohammadimehr 2020, Rahmani *et al*. 2020).

Besides, to having the projected structure to serve all the engineering purposes, in addition to the properties of the selected material, the theory to be used during the modeling stage is also of great importance. Therefore, several beam theories are developed; Euler-Bernoulli Beam Theory (EBBT), Rayleigh Beam Theory (RBT), Shear Beam Theory (SBT), and Timoshenko Beam Theory (TBT) are only four of these theories. EBBT (sometimes called the classical beam theory) is the first known beam theory as well as the most used one in the open literature due to its simplicity and efficiency. However, the theory considers only the effect of bending in the transverse vibration of the beam, and therewith the natural frequencies are overestimated and this error increases with the higher modes. RBT adds the effect of rotation of the cross-section and in this way it makes a partial correction on the values of the natural frequency of the EBBT. It should be noted that SBT and TBT are both first-order shear deformation beam theories of Timoshenko however in SBT just only shear deformation is taken into account whereas, in TBT shear

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http://www.techno-press.org/?journal=cac&subpage=8
ISSN: 1598-8198 (Print), 1598-818X (Online)

deformation and rotary inertia and their coupling effects are considered. The consideration of the effect of shear deformation in EBBT gives more accurate results for natural frequencies. Each of these four beam theory has been broadly utilized to serve different purposes in many engineering structures for many years (Timoshenko 1937, Han *et al*. 1999, Yıldırım and Kıral 2000, Wang *et al*. 2007, Civalek and Kiracioglu 2010, Civalek and Ozturk 2010, Avcar 2015, Shokravi 2017, Aydogdu *et al*. 2018, Kahya and Turan 2018, Lee and Lee 2019, Ebrahimi *et al*. 2020).

Moreover, considering the FGM properties makes the examined problem more attractive and so, solution of the free vibration problem of FG beams using different beam theories has become one of the areas of interest for the research community and addressed by many researchers (Thai and Vo 2012, Wattanasakulpong and Ungbhakorn 2012, Nguyen *et al*. 2013, Hadji *et al*. 2014, Li *et al*. 2014, Bourada *et al*. 2015, Hadji *et al*. 2015, Wang and Li, 2016, Akgöz and Civalek 2017, Al Rjoub and Hamad 2017, Avcar and Alsaid-Alwan 2017a, b, Avcar and Mohammed 2018, Ayache *et al*. 2018, Avcar 2019, Chaabane *et al*. 2019, Ramteke *et al*. 2019, Sahouane *et al*. 2019).

From the search of open literature, it is seen that a comparative study consisting of free vibration of the FG beam utilizing four different beam theories simultaneously, i.e., EBBT, RBT, SBT, TBT, has not been dealt yet. An attempt is made to address this problem in the present study. For this aim, the extension of earlier studies of authors (Avcar 2015, Avcar and Alsaid-Alwan 2017a, b) are presented. The material properties are supposed to vary continuously through the thickness direction of the beam with respect to the volume fraction of constituents. The governing equations of free vibration of FG beams are derived in the frameworks of four beam theories. Resulting equations are solved versus simply supported boundary conditions, analytically. To verify the results, comparisons are carried out with the available results. Parametrical studies are performed for discussing the effects of assumed beam theory, the variation of beam characteristics, and FGM properties on the free vibration of beams. In conclusion, it is found that the interaction between FGM properties and the supposed beam theory is of significance in terms of free vibration of the beams and that different beam theories need to be used depending on the characteristics of the beam in question.

2. Formulation of the problem

2.1 Functionally graded materials

Consider a FG beam consist of ceramic-metal, which has length, L, width b, and thickness, h, as shown in Fig. 1.

The material properties of the FG beam, namely Young's modulus E and mass density ρ, vary continuously through the thickness direction according to a function of the volume fractions of the constituents while Poisson's ratio v is taken to be constant.

Using the rule of mixture, the material properties, P, can be expressed as

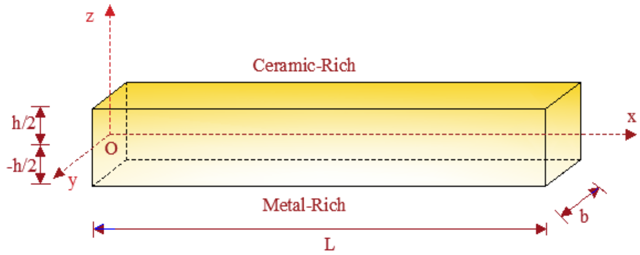

Fig. 1 Geometry of a FG beam

$$P = P_m V_m + P_c V_c \quad (1)$$

where P_m, P_c, V_m and V_c are the material properties and the volume fractions of the metal and the ceramic constituents respectively.

The total volume fraction of the metal and ceramic as follows

$$V_m + V_c = 1 \quad (2)$$

The power law of volume fraction of the ceramic constituent of the beam as follows

$$V_c = \left(\frac{z}{h} + \frac{1}{2}\right)^d \quad (3)$$

where d is a positive number ($0 \le d \le \infty$) called power law or volume fraction index, and z is the distance from the mid-plane of the beam. Note that, FG beam transforms to a fully ceramic one as d=0, while it transforms to a fully metallic one as $d=\infty$.

In the present work, unless otherwise indicated, FGM beams are supposed to composed of Aluminum (Al) and Alumina (Al_2O_3) as metal and ceramic constituents respectively, whose material properties are

$$E_m = 70 \text{ GPa}; \ \rho_m = 2702 \text{ kg/m}^3; \quad E_c = 380 \text{ GPa}; \ \rho_c = 3960 \text{ kg/m}^3 \quad (4)$$

Fig. 2 illustrates the change in Young's modulus and density of FG beams through the thickness direction versus power law index.

2.2 Governing equations

The displacements at arbitrary point of a FG beam can be expressed as

$$u(x,z,t) = u_0(x,t) + z\,\theta(x,t) \quad (5)$$

$$w(x,z,t) = w_0(x,t) \quad (6)$$

here $u_0(x,t)$ and $w_0(x,t)$ denote displacements of any point on neutral axis along the x and y directions, respectively, θ is the rotation of the cross section and t is the time.

Using the displacement field given in Eqs. (5)-(6), the normal and shear strains, can be expressed as, respectively

$$\varepsilon_x = \frac{\partial u}{\partial x} + z\frac{\partial \theta}{\partial x} \quad (7)$$

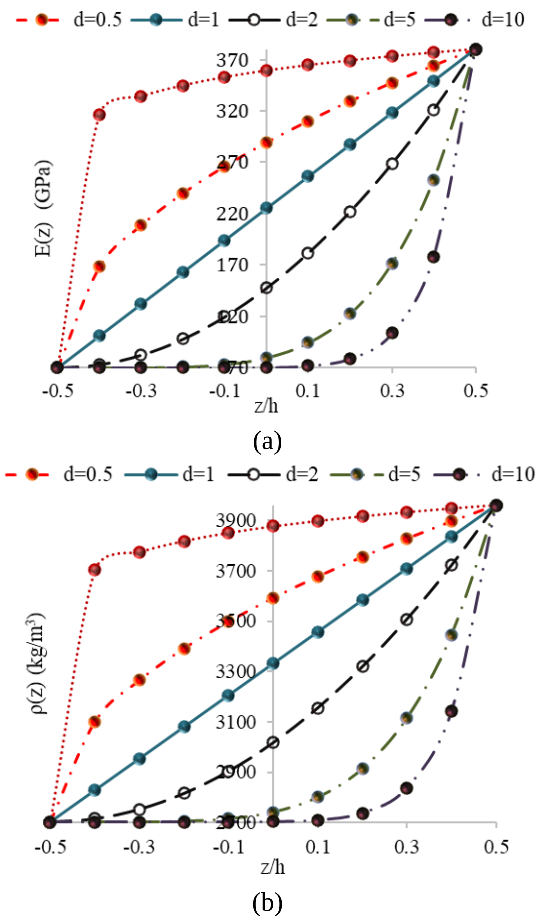

Fig. 2 The change in (a) Young's modulus and (b) density of FG beams through the thickness direction versus power law index

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \theta \tag{8}$$

Using Hooke's law, the normal and shear stresses can be expressed as, respectively

$$\sigma_x = Q_{11}\varepsilon_x \tag{9}$$

$$\tau_{xz} = Q_{55}\gamma_{xz} \tag{10}$$

here Q_{11} and Q_{55} are stiffness constants and are defined as follow

$$Q_{11} = E(z)\,,\quad Q_{55} = \frac{E(z)}{2(1+\nu)} \tag{11}$$

Note that, in this study beams with small width (b/h<0.2) are taken into consideration and so the term $1/(1-\nu^2)$ is ignored in stiffness constant Q_{11} owing to it has negligible effect.

The stress resultants can be expressed as

$$N = A_{11}\frac{\partial u}{\partial x} + B_{11}\frac{\partial \theta}{\partial x} \tag{12}$$

$$M = B_{11}\frac{\partial u}{\partial x} + D_{11}\frac{\partial \theta}{\partial x} \tag{13}$$

$$Q_x = kG_{55}\left(\frac{\partial w}{\partial x} + \theta\right) \tag{14}$$

where k is shear correction factor which is taken to be 5/6 and A_{11}, B_{11}, D_{11} and G_{55} are the material stiffness constants, which are defined as follows

$$\left(A_{11}, B_{11}, D_{11}\right) = \int_A Q_{11}\left(1, z, z^2\right) dA \tag{15}$$

$$G_{55} = \int_A Q_{55}\, dA \tag{16}$$

Using above given expressions after some mathematical operations and simplifications the governing equation of free vibration of FG beam versus EBBT, RBT, SBT and TBT found as follows, respectively

$$\lambda_{11}\frac{\partial^4 w_0}{\partial x^4} + I_0\frac{\partial^2 w_0}{\partial t^2} = 0 \tag{17}$$

$$\lambda_{11}\frac{\partial^4 w_0}{\partial x^4} + I_0\frac{\partial^2 w_0}{\partial t^2} - \xi_{11}\frac{\partial^4 w_0}{\partial x^2 \partial t^2} = 0 \tag{18}$$

$$\lambda_{11}\frac{\partial^4 w_0}{\partial x^4} + I_0\frac{\partial^2 w_0}{\partial t^2} - I_0\frac{\lambda_{11}}{kG_{55}}\frac{\partial^4 w_0}{\partial x^2 \partial t^2} = 0 \tag{19}$$

$$\lambda_{11}\frac{\partial^4 w_0}{\partial x^4} + I_0\frac{\partial^2 w_0}{\partial t^2} - \left(\xi_{11} + \frac{\lambda_{11} I_0}{kG_{55}}\right)\frac{\partial^4 w_0}{\partial x^2 \partial t^2} + \frac{\xi_{11} I_0}{kG_{55}}\frac{\partial^4 w_0}{\partial t^4} = 0 \tag{20}$$

here following definitions apply

$$\lambda_{11} = \left(D_{11} - \frac{B_{11}^2}{A_{11}}\right),\quad \xi_{11} = \left(I_2 - \frac{I_1^{\,2}}{I_0}\right) \tag{21}$$

where I_0, I_1 and I_0 are the moment of inertia components of FG beam and defined as follow

$$\left(I_0, I_1, I_2\right) = \int_A \rho(z)\left(1, z, z^2\right) dA \tag{22}$$

3. Solution of the problem

The solution of the governing Eqs. (17) and (20) can be obtained by separation of variables technique. In this case, one assumes the solution in the following form

$$w(x,t) = \alpha(x)\beta(t) \tag{23}$$

where $\alpha(x)$ is a space-dependent function, and $\beta(t)$ is temporal function, and are defined as

$$\begin{gathered}\alpha(x) = d_1\sinh(\delta x) + d_2\cosh(\delta x) + d_3\sin(\delta x) + d_4\cos(\delta x)\\ \beta(t) = c_1\cos\omega t + c_2\sin\omega t\end{gathered} \tag{24}$$

where, c_1, c_2, d_1, d_2, d_3, and d_4 are constants.

FG beam is assumed to have simply supported boundary conditions in both ends, hence these conditions can be stated in terms of w as

$$w(0,t) = 0,\quad w(L,t) = 0 \tag{25}$$

Table 1 The first three dimensionless frequencies of FG beam versus varying power law index

Mode	Results	d				
		0	0.5	1	2	5
1	Present	5.483	4.669	4.221	3.852	3.668
	Wattanasakulpong and Ungbhakorn (2012)	5.483	4.669	4.221	3.852	3.668
2	Present	21.933	18.676	16.884	15.407	14.670
	Wattanasakulpong and Ungbhakorn (2012)	21.933	18.676	16.884	15.407	14.670
3	Present	49.350	42.021	37.989	34.667	33.007
	Wattanasakulpong and Ungbhakorn (2012)	49.350	42.021	37.989	34.667	33.007

$$\frac{d^2w}{dx^2}(0,t)=0,\ \frac{d^2w}{dx^2}(L,t)=0 \tag{26}$$

Substituting Eqs. (23) and (24) into Eqs. (17)-(20) and then considering the Eqs. (25) and (26) in the obtained equation, the frequency equations of FG beam are found as follows for EBBT, RBT, SBT and TBT, respectively

$$\lambda_{11}\left(\frac{n\pi}{L}\right)^4-\omega^2 I_0=0 \tag{27}$$

$$(\lambda_{11})\left(\frac{n\pi}{L}\right)^4-\omega^2\left[I_0+\xi_{11}\left(\frac{n\pi}{L}\right)^2\right]=0 \tag{28}$$

$$\lambda_{11}\left(\frac{n\pi}{L}\right)^4-\omega^2\left[I_0+\left(\frac{\lambda_{11}I_0}{kG_{55}}\right)\left(\frac{n\pi}{L}\right)^2\right]=0 \tag{29}$$

$$\lambda_{11}\left(\frac{n\pi}{L}\right)^4-\omega^2\left\{I_0+\left[\xi_{11}+\frac{\lambda_{11}I_0}{kG_{55}}\right]\left(\frac{n\pi}{L}\right)^2\right\}+\frac{\xi_{11}I_0}{kG_{55}}\omega^4=0 \tag{30}$$

where n denotes the number of the mode.

4. Numerical results and discussion

In this section numerical examples are given to examine the present problem. At first, comparisons have been performed to show the accuracy of the present formulations. Then, parametrical studies are performed for discussing the effects of supposed beam theory, the variation of beam characteristics, and FGM properties on the free vibration of beams.

4.1 Comparison studies

Table 1 offers comparison of the first three dimensionless frequencies of FG beams with varying power law index (d) with the results of Wattanasakulpong and Ungbhakorn (2012) using EBBT. Here following material and beam properties are considered E_m=70 GPa, ρ_m=2702 kg/m^3, E_c=380 GPa, ρ_c=3960 kg/m^3 and dimensionless frequency is defined as $\Omega=\omega L^2/h\sqrt{\rho_m/E_m}$. It is apparent from Table 1 that the present results are in good agreement with the previously published ones.

Table 2 The first three dimensionless frequencies of homogenous beam

Mode	ω (rad/s)	
	Present Study	Rao (2007)
1	696.583	696.599
2	2713.365	2713.422
3	5857.951	5858.065

Table 3 The first three dimensionless frequencies of FG beam versus varying power law index

Mode	Results	d				
		0	0.5	1	2	5
1	Present	0.5414	0.4617	0.4176	0.3810	0.3620
	Al-Rjoub and Hamad (2017)	0.5414	0.4750	0.4466	0.4231	0.3967
2	Present	2.0888	1.7877	1.6199	1.4774	1.3953
	Al-Rjoub and Hamad (2017)	2.0888	1.7877	1.6199	1.4774	1.3953
3	Present	4.4480	3.8270	3.4760	3.1687	2.9674
	Al-Rjoub and Hamad (2017)	4.4480	3.8376	3.5000	3.2043	2.9953

Table 4 The first three dimensionless frequencies of FG beam versus varying power law index

Mode	Results	d				
		0	0.5	1	2	5
1	Present	5.4603	4.6511	4.2054	3.8374	3.6515
	Nguyen *et al*.(2013)	5.4603	4.6504	4.2051	3.8368	3.6509
2	Present	21.5732	18.3954	16.6398	15.181	14.4216
	Nguyen *et al*.(2013)	21.5732	18.3912	16.6344	15.1715	14.4110
3	Present	47.5921	40.6483	36.7931	33.5586	31.7972
	Nguyen *et al*.(2013)	47.5921	40.6335	36.7673	33.5135	31.7473

Table 2 presents the comparison of the first three of natural frequencies of the homogeneous beam, ω (rad/s) with results of Rao (2007) using RBT. Here the following beam and material properties are taken into account

L=1 m, b=0.05 m, h=0.15 m, d=0, E=207×10^9 Pa, ρ=76.5×10^3 N/m^3.

Table 3 shows comparison of the first three dimensionless frequencies of FG beams with varying power law index (d) with the results of Al-Rjoub and Hamad (2017) using SBT. Here following material and beam properties are considered

E_m=70 GPa, ρ_m=2702 kg/m^3, E_c=380 GPa, ρ_c=3960 kg/m^3, L/h=10

and dimensionless frequency is defined as $\Omega=\omega L\sqrt{\rho_m/E_m}$.

Table 4 demonstrates comparison of the first three dimensionless frequencies of FG beams with varying power law index (d) with the results of Nguyen *et al*. (2013) using TBT. Here following material and beam properties are considered

E_m=70 GPa, ρ_m=2702 kg/m^3, E_c=380 GPa, ρ_c=3960 kg/m^3, L/h=20

and dimensionless frequency is defined as $\Omega=\omega L^2/h\sqrt{\rho_m/E_m}$.

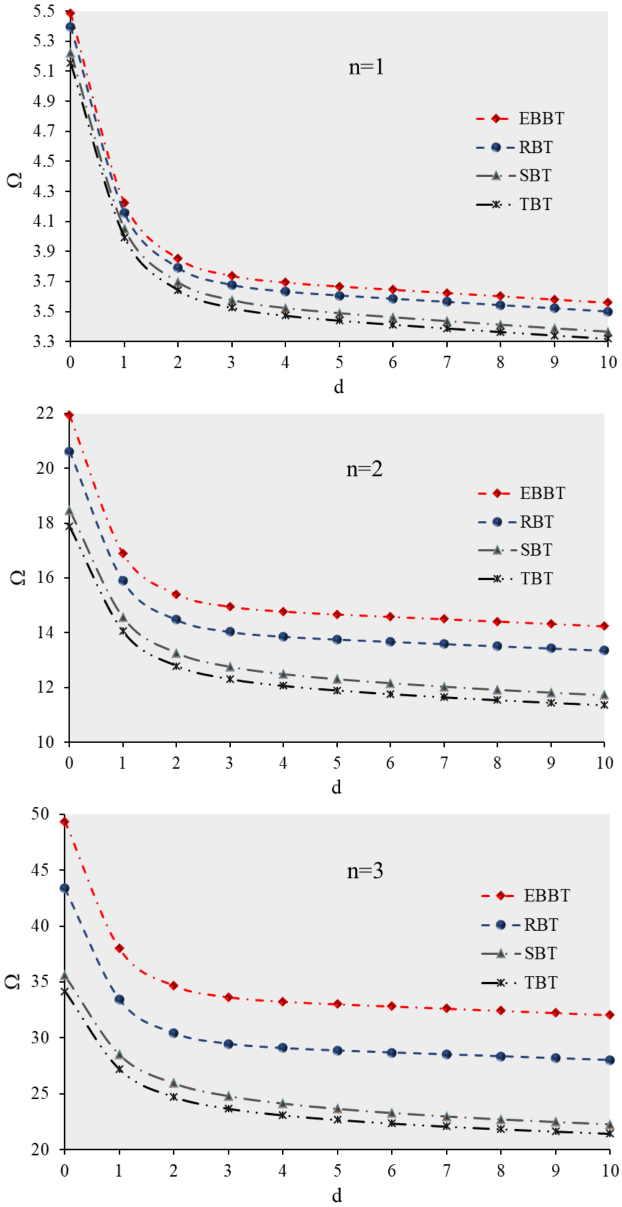

Fig. 3 The variation of first three dimensionless frequencies of FG beams versus power law index

4.2 Numerical studies

The effect of the variation of the power law index, *d*, on the dimensionless frequency values of the FG beam, $\Omega_i(i = EBBT, RBT, SBT, TBT)$ were analyzed with respect to four beam theories in Fig. 3, here *L/h*=5 was held constant. From this example, it was found that the effect of the variation of power law index, *d*, on the dimensionless frequency values vary depending on the beam theory adopted. The difference in question is lower in the fundamental mode while it increases with the increasing number of modes. Nevertheless, it was found that the effect of the variation of power law index, *d*, is higher for dimensionless frequency values obtained using EBBT and RBT when *d* coefficient values were lower, i.e., when the material used was enriched ceramic; while the change in *d* coefficient is higher for dimensionless frequency values obtained using SBT and TBT when *d* coefficient values were higher, i.e., when the material used was enriched metal, in higher number of modes.

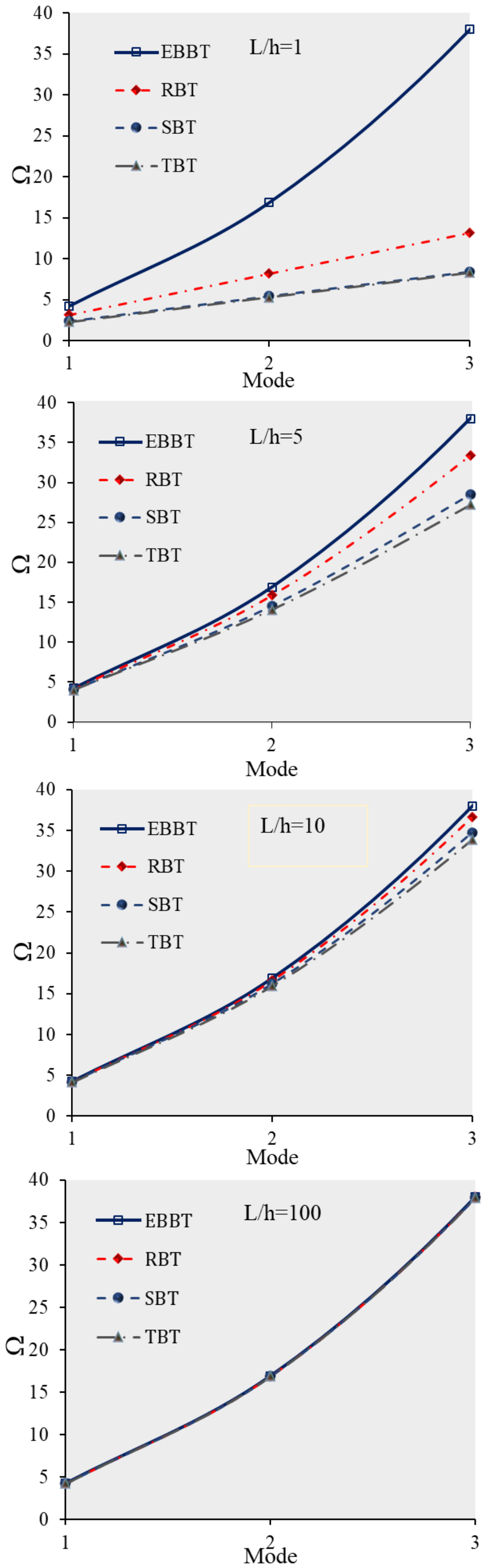

Fig. 4 The variation of dimensionless frequencies of FG beam versus span to depth ratio

The effect of the variation of the span to depth ratio, *L/h*, on the dimensionless frequency values of the FG beam,

$\Omega_i(i = EBBT, RBT, SBT, TBT)$, were analyzed with respect to four beam theories in Fig. 4, here *d*=1 was held constant. From the results obtained, it was found that the effect of the variation of the ratio, *L/h*, on dimensionless frequency values vary irregularly depending on the beam theory adopted. The effect in question is lower in the fundamental mode while it increases with the increasing number of modes. Moreover, it was concluded that the theory used gains importance in lower values of the ratio, *L/h*, i.e., in thick beams, while it was insignificant in higher values of the ratio, *L/h*, i.e., in slender beams, and that the results obtained using all four theories converge to those obtained using EBBT.

5. Conclusions

In the present work, the comparison of the supposed engineering theory for free vibration of FG beam is examined, for this aim, Euler-Bernoulli, Rayleigh, Shear and Timoshenko beam theories are employed. The FGM properties are assumed to vary continuously through the thickness direction of the beam with respect to the volume fraction of constitutes. The governing equations of free vibration of FG beams are derived in the frameworks of all engineering theories. Resulting equations are solved versus simply supported boundary conditions. To verify the results, comparisons are performed with available results. Parametrical studies are performed for discussing the effects of supposed engineering theory, the variation of beam characteristics and material properties on the free vibration of beam.

In sum, the following results were obtained:

- Dimensionless frequency values decrease when the power law index increase
- The effect of the variation of the power law index on the dimensionless frequency values varies depending on the beam theory supposed
- The effect of the variation of power law index on the dimensionless frequency values is lower in the fundamental mode while it increases with the increasing number of modes
- The effect of the variation of power law index is higher for dimensionless frequency values obtained using EBBT and RBT when the material used was enriched ceramic; while the effect of the change in power law index is higher for dimensionless frequency values obtained using SBT and TBT when the material used was enriched metal, in higher number of modes.
- The effect of the variation of the span to depth ratio on the dimensionless frequency values varies irregularly depending on the beam theory adopted. The effect in question is lower in the fundamental mode while it increases with the increasing number of modes
- The engineering theory supposed gains importance in lower values of span to depth ratio, while it is insignificant in higher values of span to depth ratio, and the results obtained using all four theories converge to those obtained using EBBT.

In conclusion, the basic beam theories are introduced, in the present study. In the end, it is found that the interaction between FGM properties and the supposed beam theory are of significance in terms of free vibration of the beams and that different theories need to be used depending on the characteristics of the beam in question. In future studies, the solutions developed in this study will be extended to other types of materials and structures in macro/micro dimensions as well as higher-order theories.

Acknowledgments

The financial support of the Suleyman Demirel University Scientific Research Projects Unit (SDU-BAP) with Grand No. 4857-YL1-17 is gratefully acknowledged. The authors would like to thank institution.

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