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Analysis of axisymmetric fractional vibration of an isotropic thin disc in finite deformation

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Abstract. This study investigates axisymmetric fractional vibration of an isotropic hyperelastic semi-linear thin disc with a view to examine effects of finite deformation associated with the material of the disc and effects of fractional vibration associated with the motion of the disc. The generalized three-dimensional equation of motion is reduced to an equivalent time fraction one-dimensional vibration equation. Using the method of variable separable, the resulting equation is further decomposed into second-order ordinary differential equation in spatial variable and fractional differential equation in temporal variable. The obtained solution of the fractional vibration problem under consideration is described by product of one-parameter Mittag-Leffler and Bessel functions in temporal and spatial variables respectively. The obtained solution reduces to the solution of the free vibration problem in literature. Finally, and amongst other things, the Cauchy's stress distribution in thin disc under finite deformation exhibits nonlinearity with respect to the displacement fields whereas in infinitesimal deformation hypothesis, these stresses exhibit linear relation with the displacement field.

Keywords: axisymmetric; finite deformation; fractional vibration; thin disc

1. Introduction

Many three-dimensional field problems in engineering and applied mathematics exhibit symmetric about the axis of rotation. Such problems, known as axisymmetric problems can be solved using one-dimensional or two-dimensional models, which are mostly described in cylindrical coordinates (Hutton 2004). The required conditions for a problem to be one-dimensional axisymmetric are that the problem domain must possess an axis of symmetry, which is conventionally taken along the longitudinal axis, that is, the domain is geometrically a solid of revolution; and all material properties, boundary conditions, and loading condition must be independent of the circumferential and longitudinal axes (Hutton 2004). It is well-known that whenever these conditions are satisfied, the deformation field (or displacement field) becomes a function of the radial coordinate and time only. Consequently, the three-dimensional elasticity equation of state naturally reduces to an equivalent one-dimensional governing equation. Thin discs are structural components with wide applications in flywheels, rotors, steams and gas turbines to mention a few. Depending on the purpose of applications, thin discs are made of concrete, steel, composite, wood, and fabricated materials. In engineering such as civil, mechanical, marine, and aeronautical, the vibration analysis of thin bodies is an important aspect of structural investigation; and an accurate analysis of stress-strain distribution and the frequencies of vibration is of much importance for the control of resonance effect thus ensuring safety of designs and structures (Senjanovic *et al*. 2015, Burago *et al*. 2017, Das *et al*. 2010, Deng *et al*. 2010, Fadodun *et al*. 2017a, 2017b, Lychev 2011, Zur 2015, 2016a, 2016b, Bennoue 2016, Berferhal *et al*. 2016). In fact, the study of vibration of thin discs is a subject of considerable scientific and practical interests which has been investigated extensively, and is still receiving significant attention (Das *et al*. 2010, Deng *et al*. 2010). Lyu *et al.* (2017) used improved Fourier series method together with the Rayleigh-Ritz procedure to investigate free in-plane vibration analysis of elastically restrained annular panels made of functionally graded materials. Based on the proposed approach, the influence of important parameter such as FGM power-law exponent and boundary restraints were examined on the in-plane vibration of the panels via numerical results. Zur (2017) analyzed free vibration of elastically supported functionally graded circular plates using quasi-Green's method. In the study, he presented the effects of boundary conditions, volume fraction index, and the stiffness of ring supports on natural frequencies which were ignored in the previous studies. Zhong *et al.* (2018) employed newly proposed semi-analytical procedure for vibration analysis of functionally graded carbon nanotube reinforced composites (FG-CNTRC) circular, annular and sector plates. With the aid of Ritz-variational energy method, the plates natural frequencies and their accompanying mode shapes were obtained. Furthermore, the convergence, accuracy, stability, and efficiency of the computational model were examined and the influence of geometrical parameters, CNTs distributions, volume fraction of CNTs and boundary conditions were presented. Hasheminejad *et al*. (2013) formulated a two-dimensional analytical model and

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provided exact solution for the free extensional vibration of an elastic elliptical thin plate. Bouboulas and Anifantis considered vibration analysis of a rotating disk with crack. Zur (2015, 2016a, 2016b) used Green's function approach to investigate frequency analysis of thin annular plates with nonlinear variable thickness. Gorman *et al*. (2001) studied the effect of structural-acoustic coupling using analytical-numerical approach in the free vibration analysis of a thin disk interacting with an acoustic medium in a cylindrical duct. Jaroszewicz (2017) investigated natural frequencies of homogeneous and isotropic circular thin plates with nonlinear thickness variation and clamped edge conditions. Das *et al*. (2010) employed energy principle and presented the out-of-plane free vibration analysis of a rotating annular disc under the application of uniform transverse pressure. Burago *et al.* (2016) studied stress-strain distribution state in stationary vibration and fatigue failure of compressor disk of variable thickness. Batra and Iaccarino (2008) obtained exact solution for radial deformation of functionally-graded isotropic and incompressible second order elastic cylinder. Bashmal *et al*. (2010) investigated in-plane vibration of circular annular disc and obtained frequency equation for various types of boundary conditions. Sharmal *et al.* (2012) employed finite element analysis to investigate displacement, strain, and stress distributions in a thin circular functionally graded thermo-elastic disks subjected to thermal loads. Ursoniu *et al.* (2017) examined the influenced of stiffening ribs on natural frequencies of butterfly valve disk. Kumar *et al.* (2017) investigated effects of time and diffusion phase-lags in a thin circular discs with axisymmetric heat supplied. Zhang *et al.* (2017) studied the vibration localization analysis of mistuned bladed disk system. Deng (2010) studied vibration of spinning discs and powder formation in centrifugal atomization. It is well-known that fractional derivative is a generalization of the usual integer order derivative. In recent time, researchers in physics, applied mathematics, and engineering are using fractional derivatives to investigate and apprehend effects associated with anomalous processes in media (Fu *et al*. 2013, Li 2014, Fadodun *et al*. 2017b). For instance, Treeby *et al*. (2010) used fractional Laplacian to model power law absorption and dispersion of acoustic wave propagation in media. Chen *et al*. (2010) presented solution of fractional diffusion equation by Kansa method. Li (2014) constructed analytic solution of a fractional generalized two phase Lame-Clapeyron Stefan problem. Fu *et al.* (2013) used boundary particle approach for the Laplace transform time-fractional diffusion equation. Fadodun *et al*. (2017b) presented exact solution of fractional vibration problem of radially vibrating non-classical cylinder. Du *et al*. (2010) presented a compact difference scheme for fractional diffusion wave equation. It has been established that finite deformation investigation of solid bodies within the elastic regime provides the avenue to apprehend effects which the infinitesimal deformation theory often fails to capture (Akinola 1999, 2001, Ciarlet 1998). For instance, an elastic body in simple shear under infinitesimal deformation exhibits only shear stresses, whereas in finite deformation approach, the body exhibits Ponyting effect, that is, the presence of both shear and normal stresses (Akinola 1999). Furthermore, a bar in torsion under finite deformation exhibits Kelvin effect, that is, the presence of axial stresses which are not captured in the domain of infinitesimal deformation theory (Akinola 2001). In this study, we consider axisymmetric fractional vibration of thin disc in finite deformation. The disc is made of an isotropic semi-linear hyperelastic John's material (Fadodun and Akinola 2017a). In all of the above mentioned studies, the problems concerned with the fractional (anomalous) vibration of non-classical thin disc under finite deformation has not been investigated. Therefore, this work focuses on anomalous vibration of axisymmetric thin circular disc made of semilinear hyperelastic John's material. The aims of this study is to investigate effects associated with finite deformation and fractional vibration of the disc under consideration. In the present work, the Caputo fractional derivative is employed to model anomalous process in temporal variable due to its amenability for solving initial-boundary value problems. Amongst other things, it is obtained that the non-zero components of Cauchy's stress distribution in the disc exhibit nonlinearity with respect to the displacement fields. Furthermore, the time-fractional effect of the motion of the disc is characterized by the one-parameter Mittag-Leffler function. The results in this work find applications in the anomalous vibration analysis of flywheels, rotors, steams and gas turbines. The rest of the paper is organized as follow: section two gives statement of the problem and the geometry of deformation, section three details the generalized equation of motion, section four presents the initial-boundary value problem, section five gives the solution of the governing initial-boundary value problem, section six presents the stress distribution in the disc, section seven presents the graphical illustration while section eight concludes the study.

2. Statement of the problem and deformation geometry

2.1 Statement of the problem

Consider axisymmetric anomalous vibration of an isotropic homogeneous thin disc in finite deformation. The disc which is made of hyperelastic semi-linear John's material occupies a region Ω of subset of the three-dimensional space ($\Omega \subset \Re^3$). The disc has inner and outer radii $r=a$ and $r=b$, $b>a$, and the deformation of the disc from reference configuration Ω_0 onto a current configuration Ω takes the form

$$R = R(r,t)\,,\;\; \varphi = \theta\,,\quad Z = \delta z\,, \tag{1}$$

where the constant $\delta \in \Re$, $\delta \neq 0$, the coordinates (r,θ,z) denotes the material points of the disc in the reference configuration Ω_0 and the coordinates (R,φ,Z) represents the material coordinates of the same point in the current configuration Ω.

2.2 Geometry of deformation

Let the vectors $\vec{e}_r, \vec{e}_\theta, \vec{e}_z$ and $\vec{e}_R, \vec{e}_\varphi, \vec{e}_Z$ denote the orthonormal base vectors in reference and current configurations Ω_0 and Ω respectively, then in view of Eq. (1), the vectors $\vec{r}$ and $\vec{R}$ which give the position vectors of material points in the reference and current configurations Ω_0 and Ω assume the form (Fadodun *et al*. 2017c)

$$\vec{r} = r\vec{e}_r + z\vec{e}_z, \quad \vec{R} = R(r,t)\vec{e}_R + Z\vec{e}_Z. \tag{2}$$

Now, let the geometry of deformation of the disc from the reference configuration Ω_0 onto current configuration Ω be the gradient of deformation $\tilde{F} = \nabla\vec{R}$, where ∇ is the gradient operator in the reference configuration Ω_0 (Fadodun *et el*. 2017b, 2017c).

Using Eq. (2), the gradient of deformation $\tilde{F} = \nabla\vec{R}$ is

$$\tilde{F} = \frac{\partial R}{\partial r}\vec{e}_r \otimes \vec{e}_R + \frac{R}{r}\vec{e}_\theta \otimes \vec{e}_\varphi + \delta\vec{e}_z \otimes \vec{e}_Z. \tag{3}$$

Using polar decomposition theorem, there exists unique left symmetric positive definite stretch tensor $\tilde{U}$ and orthogonal rotation tensor $\tilde{O}^D$ such that (Fadodun *et al*. 2017c)

$$\tilde{F} = \tilde{U}\tilde{O}^D. \tag{4}$$

The tensors $\tilde{U}$ and $\tilde{O}^D$ in Eq. (4) satisfy the equations

$$\tilde{U}^2 = \tilde{F}\tilde{F}^T, \quad \tilde{O}^D = \tilde{U}^{-1}\tilde{F} \tag{5}$$

where $\tilde{F}^T$ is the transpose of the gradient of deformation $\tilde{F}$ and $\tilde{U}^{-1}$ is the inverse of the tensor $\tilde{U}$.

The combination of Eq. (4) and Eq. (5) gives

$$\tilde{U} = \tilde{F}, \quad \tilde{O}^D = \tilde{E} \tag{6}$$

where $\tilde{E} = \vec{e}_r \otimes \vec{e}_R + \vec{e}_\theta \otimes \vec{e}_\varphi + \vec{e}_z \otimes \vec{e}_Z$ is the rank-two unit tensor.

3. Generalized equation of motion

The energy function for an isotropic homogeneous hyperelastic material (Fadodun and Akinola 2017a) is

$$W = \mu I_1(\tilde{U} - \tilde{E})^2 + \frac{\lambda}{2} I_1^2(\tilde{U} - \tilde{E}), \tag{7}$$

where $I_1(\tilde{U} - \tilde{E})$ is the first invariant of the tensor $(\tilde{U} - \tilde{E})$ and λ, μ are the Lame constants of the material of the disc.

Let the first Piola-Kirchhoff's stress tensor $\tilde{P}$ be energy conjugate to the gradient of deformation $\tilde{F}$, then by Frechet derivative

$$\tilde{P} = \frac{\partial}{\partial \tilde{F}} W. \tag{8}$$

Substituting Eq. (7) into Eq. (8) gives

$$\tilde{P} = 2\mu\tilde{F} + (\lambda I_1(\tilde{U} - \tilde{E}) - 2\mu)\tilde{O}^D. \tag{9}$$

In view of Eq. (9), the time-fractional motion equation governing the anomalous vibration of the thin disc is

$$\nabla \cdot \tilde{P} = \frac{\rho}{\xi^{2-\beta}} \frac{\partial^\beta}{\partial t^\beta} R, \quad 1 < \beta \le 2 \tag{10}$$

where ρ is the material density of the thin disc, and the parameter ζ having dimension of time is introduced to ensure all terms in Eq. (10) are dimensionally balanced.

4. Initial-boundary value problem

Let P_{RR}, $P_{\varphi\varphi}$, P_{ZZ} and $P_{R\varphi}$, $P_{\varphi R}$, P_{RZ}, P_{ZR}, $P_{Z\varphi}$, $P_{\varphi Z}$ denote the normal and share components of the first Piola-Kirchhoff's tensor $\tilde{P}$.

Using Eqs. (3) and (6) in Eq. (9) yields

$$P_{RR} = 2\mu\left(\frac{\partial R}{\partial r} - 1\right) + \lambda\left(\frac{\partial R}{\partial r} + \frac{R}{r} + \delta - 3\right), \tag{11a}$$

$$P_{\varphi\varphi} = 2\mu\left(\frac{R}{r} - 1\right) + \lambda\left(\frac{\partial R}{\partial r} + \frac{R}{r} + \delta - 3\right), \tag{11b}$$

$$P_{zz} = 2\mu(\delta - 1) + \lambda\left(\frac{\partial R}{\partial r} + \frac{R}{r} + \delta - 3\right), \tag{11c}$$

and

$$P_{R\varphi} = P_{\varphi R} = P_{RZ} = P_{ZR} = P_{\varphi Z} = P_{Z\varphi} = 0 \tag{11d}$$

respectively.

Invoking the hypothesis of plane stress P_{ZZ}=0 yields

$$\delta = 1 - \frac{\lambda}{2\mu + \lambda}\left(\frac{\partial R}{\partial r} + \frac{R}{r} - 2\right). \tag{12}$$

Substituting Eq. (12) into Eqs. (11a) and (1b) gives

$$P_{RR} = 2\mu\left(\frac{\partial R}{\partial r} - 1\right) + \frac{2\mu\lambda}{2\mu + \lambda}\left(\frac{\partial R}{\partial r} + \frac{R}{r} - 2\right), \tag{13a}$$

$$P_{\varphi\varphi} = 2\mu\left(\frac{R}{r} - 1\right) + \frac{2\mu\lambda}{2\mu + \lambda}\left(\frac{\partial R}{\partial r} + \frac{R}{r} - 2\right). \tag{13b}$$

Knowing that only components P_{RR} and $P_{\varphi\varphi}$ of $\tilde{P}$ are non-zero, then Eq. (10) reduces to

$$\frac{\partial P_{RR}}{\partial r} + \frac{1}{r}(P_{RR} - P_{\varphi\varphi}) = \frac{\rho}{\xi^{2-\beta}} \frac{\partial^\beta R}{\partial t^\beta}. \tag{14}$$

Using Eqs. (13a) and (13b) in Eq. (14) gives the time-fractional partial differential equation governing the anomalous vibration of the thin disc under consideration

$$v^2\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} - \frac{R}{r^2}\right) = \frac{1}{\xi^{2-\beta}} \frac{\partial^\beta R}{\partial t^\beta}, \tag{15}$$

where $v^2 = \dfrac{4\mu(\mu + \lambda)}{(2\mu + \lambda)\rho}$, v is the wave speed, and $1<\beta\le 2$.

In view of Eq. (15) and introducing a parameter α, such that $\beta=2\alpha$, the governing initial-boundary value problem of the disc under consideration reads

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} - \frac{R}{r^2} = \frac{1}{v^2 \xi^{2(1-\alpha)}} \frac{\partial^{2\alpha} R}{\partial t^{2\alpha}}, \tag{16a}$$

$$R(a,t)=0\,,\quad R(b,t)=0 \tag{16b}$$

$$R(r,0)=h(r)\,,\quad \frac{\partial R(r,0)}{\partial t}=0\,, \tag{16c}$$

where the constants a is the inner radius, b is the outer radius, $\frac{1}{2}<\alpha\le 1$, and $h(r)$ is a function of radius r only.

Remark 1: Eq. (16a) generalizes the classical motion equation of thin disc in literature. In fact, it reduces to the classical motion equation of thin disc when α=1.

5. Solution of the governing initial-boundary value problem

5.1 Highlight of the caputo derivative

This sub-section highlights the Caputo fractional differential operator of order α (Ishteva 2005, Zhang and Yang 2015).

The Caputo fractional derivative of order α, $n-1<\alpha\le n$, $n\in\aleph$ of function $f(t)$ is defined by

$$\frac{d^\alpha f(t)}{dt^\alpha}:=\begin{cases}\dfrac{d^n f(t)}{dt^n}, & \alpha=n\\ \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t \frac{f^n(T)}{(t-T)^{\alpha-n+1}}dT & n-1<\alpha<n\end{cases}. \tag{17}$$

where $n\in\aleph$, $\aleph$ is the set of natural numbers and $\Gamma(\cdot)$ denotes the Gamma function. It is obvious that for $\alpha=n$, the Caputo fractional derivative coincides with the standard derivative.

5.2 Solution of the initial-boundary value problem

Let the field variable $R(r,t)$ in Eq. (16a) assumes the form

$$R(r,t)=\Psi(r)\Phi(t)\,, \tag{18}$$

where $\Psi(r)$ is a function of r only and $\Phi(t)$ is a function of t only.

Substituting Eq. (18) into Eq. (16a) yields an ordinary differential equation

$$r^2\frac{d^2\Psi}{dr^2}+r\frac{d\Psi}{dr}+((\chi_n r)^2-1)\Psi=0\,, \tag{19}$$

and a fractional ordinary differential equation

$$\frac{d^{2\alpha}\Phi}{dt^{2\alpha}}+(v\chi_n\xi^{(1-\alpha)})^2\Phi=0\,. \tag{20}$$

The solution of Eq. (19) is

$$\Psi(r)=d_1J_1(\chi_n r)+d_2Y_1(\chi_n r)\,, \tag{21}$$

where $J_1(\chi_n r)$, $Y_1(\chi_n r)$ are the Bessel functions of order one of first and second kind respectively, and d_1, d_2 are arbitrary constants.

Using Eq. (21) and boundary conditions in Eq. (16b) gives the characteristic equation

$$\begin{pmatrix}J_1(\chi_n a) & Y_1(\chi_n a)\\ J_1(\chi_n b) & Y_1(\chi_n b)\end{pmatrix}\begin{pmatrix}d_1\\ d_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}. \tag{22}$$

Eq. (22) which involves the unknowns d_1, d_2 represents a linear homogeneous system of two equations. The condition(s) for the existence of non-zero solutions is that the determinant of the coefficient of the system should be zero. That is,

$$\begin{vmatrix}J_1(\chi_n a) & Y_1(\chi_n a)\\ J_1(\chi_n b) & Y_1(\chi_n b)\end{vmatrix}=0\,, \tag{23}$$

$$J_1(\chi_n a)Y_1(\chi_n b)-J_1(\chi_n b)Y_1(\chi_n a)=0\,. \tag{24}$$

The solution χ_n of Eq. (24) are the eigenvalues (frequencies); and the corresponding associated eigenfunctions yield the arbitrary constants d_1, d_2.

It is clear that for each solution set χ_n, d_1, d_2 via Eqs. (22)-(24), one has the solution

$$\Psi(r)=d_1J_1(\chi_n r)+d_2Y_1(\chi_n r)\,. \tag{25}$$

Next, the solution of Eq. (20) in view of Eq. (17) gives

$$\Phi(t)=E_{2\alpha}((i\varpi)^2t^{2\alpha})\,, \tag{26}$$

where $\varpi=v\chi_n\xi^{(1-\alpha)}$ and in general

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}\,,\Re(\alpha)>0\,,z,\alpha\in C \tag{27}$$

is a one-parameter Mittag-Leffler function and C is the complex plane

Substituting Eqs. (25) and (26) into Eq. (18) gives

$$R=(d_1J_1(\chi_n r)+d_2Y_1(\chi_n r))E_{2\alpha}((i\varpi)^2t^{2\alpha})\,, \tag{28}$$

The combination of Eq. (28) and Eq. (16c) yields

$$h(r)=d_1J_1(\chi_n r)+d_2Y_1(\chi_n r)\,,\quad \frac{\partial R(r,0)}{\partial t}=0\,. \tag{29}$$

Clearly, the obtained solution in Eq. (28) satisfies the initial and boundary conditions in Eqs. (16b)-(16c) for the choice of function $h(r)$ defined in Eq. (29).

In the special case of α=1, the one-parameter Mittag-Leffler function $E_{2\alpha}((i\varpi)^2t^{2\alpha})$ in Eq. (26) reduces to

$$E_2((i\varpi)^2t^2)=\cos(\varpi t)\,, \tag{30}$$

and consequently, the obtained solution in Eq. (28) degenerates to

$$R=(d_1J_1(\chi_n r)+d_2Y_1(\chi_n r))\cos(\varpi t)\,. \tag{31}$$

Remark 2: The obtained solution in Eq. (28) which gives the deformation function for the fractional vibration of the disc under consideration reduces to the solution of the free vibration problem (Eq. (31)) in literature.

6. Cauchy's stress distribution

Let $\tilde{T}$ denote the Cauchy-stress tensor, the stress tensor $\widetilde{T}$ and the first Piola-Kirchhoff's stress tensor $\widetilde{P}$ are related by

$$\widetilde{T} = \frac{d\Omega_0}{d\Omega}\widetilde{F}\widetilde{P} = \frac{d\Omega_0}{d\Omega}\nabla\vec{R}\widetilde{P}\,, \tag{32}$$

where $d\Omega_0$ is the elemental volume in the reference configuration Ω_0 and $d\Omega$ is the elemental volume in the current configuration Ω.

In view of Eq. (2), the elemental volumes $d\Omega_0$ and $d\Omega$ are

$$d\Omega_0 = \frac{\partial}{\partial r}\vec{r}\cdot\left(\frac{\partial}{\partial\theta}\vec{r}\times\frac{\partial}{\partial z}\vec{r}\right) = r\,, \tag{33a}$$

$$d\Omega = \frac{\partial}{\partial r}\vec{R}\cdot\left(\frac{\partial}{\partial\theta}\vec{R}\times\frac{\partial}{\partial z}\vec{R}\right) = R\frac{\partial R}{\partial r}\,. \tag{33b}$$

Let T_{RR}, $T_{\varphi\varphi}$, T_{ZZ} and $T_{R\varphi}$, T_{RZ}, $T_{\varphi Z}$ denote the normal and shear components of the symmetric Cauchy stress tensor $\widetilde{T}$.

Substituting Eqs. (3), (33a), and (33b) into Eq. (32) gives

$$T_{RR} = \frac{r}{R(r,t)}P_{RR}\,, \text{ and } T_{\varphi\varphi} = \left(\frac{\partial R}{\partial r}\right)^{-1}P_{\varphi\varphi} \tag{34a}$$

$$T_{ZZ} = T_{RZ} = T_{R\varphi} = T_{\varphi Z} = 0 \tag{34b}$$

where P_{RR} and $P_{\varphi\varphi}$ are the non-zero components of the first Piola-Kirchhoff's stress tensor $\widetilde{P}$.

Substituting Eqs. (13a) and (13b) into Eq. (34a) gives

$$T_{RR} = \frac{r}{R}\left(2\mu\left(\frac{\partial R}{\partial r}-1\right)+\frac{2\mu\lambda}{2\mu+\lambda}\left(\frac{\partial R}{\partial r}+\frac{R}{r}-2\right)\right) \tag{35a}$$

and

$$T_{\varphi\varphi} = \left(\frac{\partial R}{\partial r}\right)^{-1}\left(2\mu\left(\frac{\partial R}{\partial r}-1\right)+\frac{2\mu\lambda}{2\mu+\lambda}\left(\frac{\partial R}{\partial r}+\frac{R}{r}-2\right)\right) \tag{35b}$$

respectively.

It is well known that the deformation function $R=R(r,t)$ and the displacement function $u=u(r,t)$ are related by

$$R(r,t) = r + u(r,t)\,. \tag{36}$$

Substituting Eq. (36) into Eqs. (35a) and (35b) gives

$$T_{RR} = \left(1+\frac{u}{r}\right)^{-1}\left(2\mu\frac{\partial u}{\partial r}+\frac{2\mu\lambda}{2\mu+\lambda}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)\right), \tag{37a}$$

$$T_{RR} = \left(1+\frac{\partial u}{\partial r}\right)^{-1}\left(2\mu\frac{\partial u}{\partial r}+\frac{2\mu\lambda}{2\mu+\lambda}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)\right). \tag{37b}$$

Eqs. (37a) and (37b) indicate that the non-zero components T_{RR} and $T_{\varphi\varphi}$ exhibit nonlinearity with respect to displacement function $u=u(r,t)$.

In view of Eqs. (37a) and (37b), the non-zero components T_{RR} and $T_{\varphi\varphi}$ are related by

$$\left(1+\frac{u}{r}\right)T_{RR} = \left(1+\frac{\partial u}{\partial r}\right)T_{\varphi\varphi} + 2\mu\left(\frac{\partial u}{\partial r}-\frac{u}{r}\right). \tag{38}$$

In literature, it is worth mentioning that under infinitesimal deformation theory, the non-zero Cauchy's stress components T^*_{RR} and $T^*_{\varphi\varphi}$ in the disc are given by

$$T^*_{RR} = 2\mu\frac{\partial u}{\partial r}+\frac{2\mu\lambda}{2\mu+\lambda}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right), \tag{39a}$$

$$T^*_{\varphi\varphi} = 2\mu\frac{u}{r}+\frac{2\mu\lambda}{2\mu+\lambda}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right). \tag{39b}$$

Furthermore, the stresses T^*_{RR} and $T^*_{\varphi\varphi}$ in Eqs. (39a) and (39b) are related by

$$T^*_{RR} = T^*_{\varphi\varphi} + 2\mu\left(\frac{\partial u}{\partial r}-\frac{u}{r}\right). \tag{40}$$

Remark 3: It is observed that in finite deformation consideration, the non-zero Cauchy's stress components T_{RR} and $T_{\varphi\varphi}$ exhibit nonlinearity with respect to the displacement field and its derivative (Eqs. (37a) and (37b)), whereas these stresses are linearly related to the displacement field and its derivative in the case of infinitesimal deformation theory (Eqs. (39a) and (39b)). Furthermore, the relationship between T_{RR} and $T_{\varphi\varphi}$ in Eq. (38) generalizes the relationship between T^*_{RR} and $T^*_{\varphi\varphi}$ in Eq. (40).

7. Graphical illustrations

For purpose of graphical illustration, we select a thin disc with density $\rho=1.2\times10^3$ m^3, inner radius a=0.5 m, outer radius b=2.0 m and Lame constants $\lambda=2.7\times10^6$ N/m^2, $\mu=3.1\times10^6$ N/m^2. It is observed that in amenable elastic thin disc undergoing fractional vibration the wave phenomenon effect increases as α approaches 1 (i.e., $\alpha\to1$).

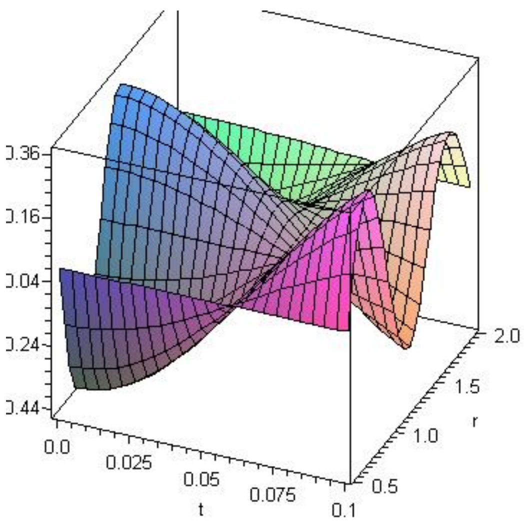

Fig. 1 Deformation function when α=1

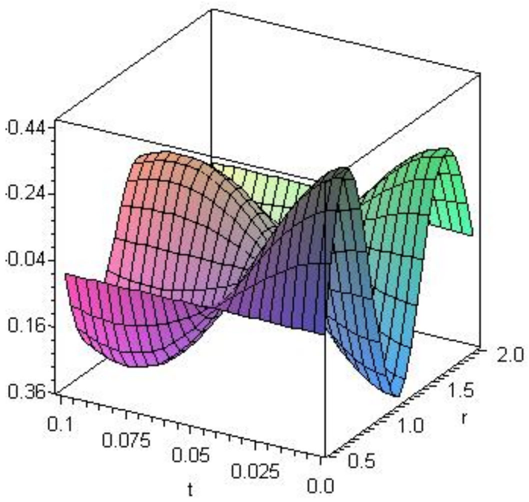

Fig. 2 Deformation function when α=0.9

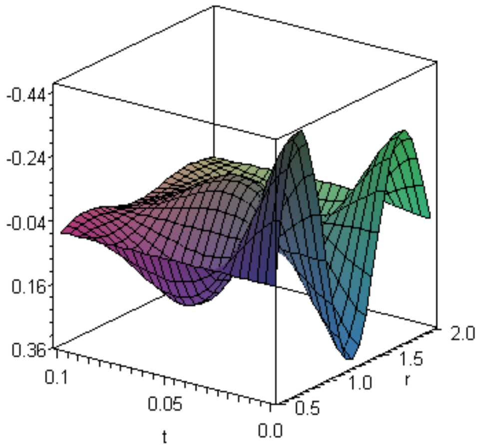

Fig. 3 Deformation function when α=0.8

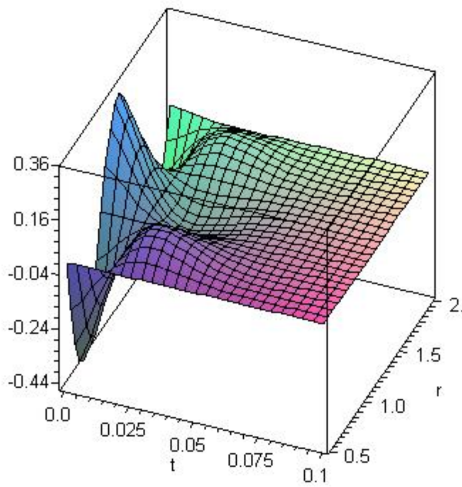

Fig. 4 Deformation function when α=0.7

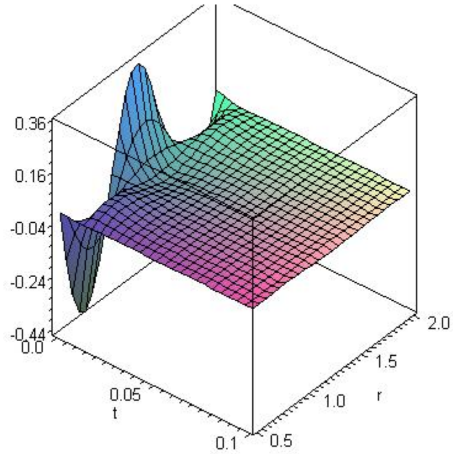

Fig. 5 Deformation function when α=0.6

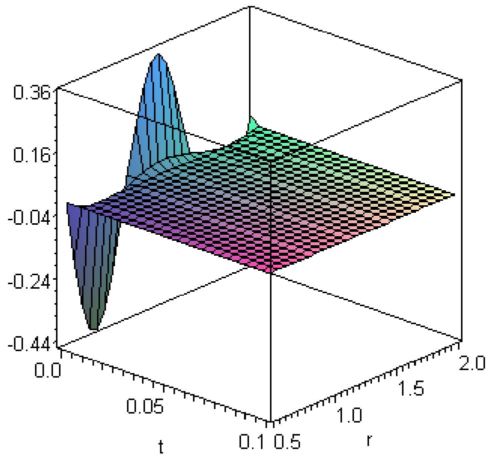

Fig. 6 Deformation function when α=0.51

8. Conclusions

The study employs Caputo fractional derivative and variable separable approach to construct solution of axisymmetric fractional vibration problem of thin disc in finite deformation. The disc is made of hyperelastic semi-linear John's material. The solution of the problem which gives the deformation distribution in the disc is expressed in terms of Mittag-Leffler and Bessel functions**.** It is observed that in amenable elastic thin disc undergoing fractional vibration the wave phenomenon effect increases as order of fractional derivative parameter α approaches 1 (that is, $\beta \to 2$). Also, the obtained solution in Eq. (28) reduces to the solution of the classical cylindrical wave equation (Eq. (31)) in literature. In addition, the stress distribution in the disc exhibits nonlinear relationship with the displacement distribution and its derivative (Eqs. (37a), (37b)). The results in this study find applications in the anomalous vibration analysis of annular thin disc, flywheel, and rotors under finite deformation.

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