Computers and Concrete, Vol. 2, No. 2 (2005) 125-146

DOI: http://dx.doi.org/10.12989/cac.2005.2.2.125

A method to evaluate the frequencies of free transversal vibrations in self-anchored cable-stayed bridges

Pietro Monaco† and Alessandra Fiore‡

Civil and Environmental Engineering Department, Polytechnic of Bari, Orabona street - 4, 70100 Bari, Italy
(Received November 7, 2003, Accepted March 25, 2005)

Abstract. The objective of this paper is setting out, for a cable-stayed bridge with a curtain suspension, a method to determine the modes of vibration of the structure. The system of differential equations governing the vibrations of the bridge, derived by means of a variational formulation in a nonlinear field, is reported in Appendix C. The whole analysis results from the application of Hamilton's principle, using the expressions of potential and kinetic energies and of the virtual work made by viscous damping forces of the various parts of the bridge (Monaco and Fiore 2003). This paper focuses on the equation concerning the transversal motion of the girder of the cable-stayed bridge and in particular on its final form obtained, restrictedly to the linear case, neglecting some quantities affecting the solution in a non-remarkable way. In the hypotheses of normal mode of vibration and of steady-state, we propose the resolution of this equation by a particular method based on a numerical approach. Respecting the boundary conditions, we derive, for each mode of vibration, the corresponding frequency, both natural and damped, the shape-function of the girder axis and the exponential function governing the variability of motion amplitude in time. Finally the results so obtained are compared with those deriving from the dynamic analysis performed by a finite elements calculation program.

Keywords: cable-stayed bridge; nonlinear analysis; variational formulation; trasversal motion; modes of vibration; numerical approach; software.

1. Introduction

This paper treats of the case of a self-anchored cable-stayed bridge, with two spans, a suspension converging to the top of the tower in correspondence with the bridge's axis, and an A-shaped pylon (Leonhardt and Zellner 1991). The hypothesis is introduced that the suspension cables are infinitely close, that is to say a "curtain" hypothesis, and that once the tower is built and the assemblage of the girder elements is finished, they will be subject to successive stretchings, so that the girder gets an almost straight configuration, when the construction is complete. The only external force acting on the girder constists of a uniform dead load g_T, therefore the prestressed state of structural elements, on the stay stretching operation, is characterized by traction in the stays, axial compression in the girder and bending in the tower (Fig. 1).

† Full Professor, E-mail: p.monaco@poliba.it
‡ PhD Student, E-mail: a.fiore@poliba.it

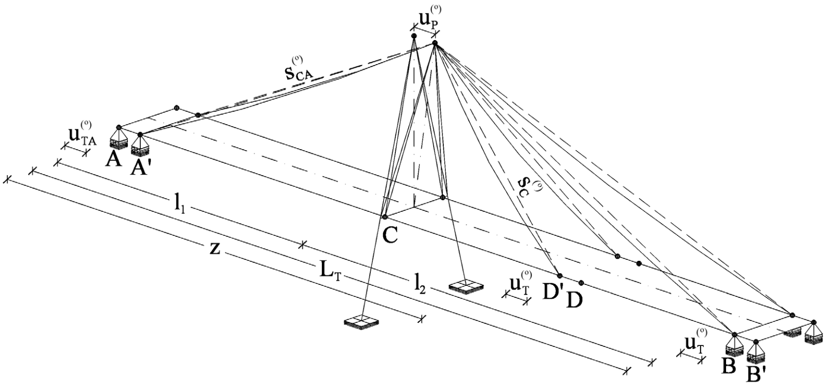

Fig. 1 Prestress configuration

As to the tower, a cantilever-like behaviour is hypothesized: it is simulated by elastic constraints and by an ideal mass concentrated on its top.

We also introduce a resistence to the motion acting on the mass with a sign opposite to that of velocity and an intensity proportional to the velocity modulus, according to a damping constant c. Damping constant c is expressed in terms of critical damping c_c:$c = \xi c_c$, where ξ represents the damping ratio. Subsequently we refer to: the area respectively of the anchorage cable and of the generic curtain cable as A_{CA}, A_C; the curtain density of the area of the stays as a_C; the elasticity modulus, the area, the mass per length unit, the inertial moments of the pylon calculated with respect to the barycentric axis along the directions x and z of the global system, as E_P, A_P, m_P, J_P, J_{Pz}; the elasticity modulus, the area, the mass per length unit, the inertial moment of the girder calculated with respect to its barycentric axis parallel to the global axis y, as E_T, A_T, m_T, J_T; the inertial moment of the two single beams constituting the girder, calculated with respect to its own barycentric axis parallel to the global axis x, as J_l; the axial forces in the initial configuration acting respectively in anchorage and curtain cables and in the girder as $S_{CA}^{(\circ)}$, $S_C^{(\circ)}$, $N_T^{(\circ)}$. The lengths of the anchorage and curtain cable spans are indicated respectively by $s_{CA}^{(\circ)}$ and $s_C^{(\circ)}$. The sag variations of the stays are neglected.

Straight cables are hypothesized, both in their initial and final configurations, yet introducing the equivalent tangent Dishinger modulus E_{CA} and E_C, respectively for the anchorage cable and the generic curtain cable, in order to take into account their real behaviour. The structural deformations are: the longitudinal displacements u_P, u_T and the transversal ones w_P, w_T respectively of the tower top and of the barycentre of a generic section of the girder; the vertical displacement v_T of the barycentre of a generic section of the girder; the vertical displacements v_{C1}, v_{C2}, the transversal displacements w_{C1} w_{C2} and the longitudinal ones u_{C1}, u_{C2} of the anchorage points of the stays to the girder. The vertical displacement of the tower top is regarded as equal to zero, so that its axial deformability is neglected. For small vibrations, putting $w_{C1} \cong w_{C2} \cong w_T$, the vertical and longitudinal displacements of the anchorage points of the cables to the girder and of the barycentre of the girder's generic section, can be expressed in terms of the transversal displacement w_T (Monaco and Fiore 2003):

$$v_T \cong \left(-\frac{w_T^2}{2h_P}\right)\cdot\left(\frac{h_P^2}{h_P^2+(z-l_1)^2}\right); \quad v_{C1} = \left(-\frac{w_T^2}{2h_P} - \frac{b_T}{h_p}w_T\right)\cdot\frac{h_P^2}{h_P^2+(z-l_1)^2};$$

$$u_{C1} = \left(-\frac{w_T^2}{2} - b_T w_T\right)\cdot\frac{z-l_1}{h_P^2+(z-l_1)^2}$$

$$u_T \cong \left(-\frac{w_T^2}{2}\right)\cdot\left(\frac{(z-l_1)}{h_P^2+(z-l_1)^2}\right); \quad v_{C2} = \left(-\frac{w_T^2}{2h_P} + \frac{b_T}{h_p}w_T\right)\cdot\frac{h_P^2}{h_P^2+(z-l_1)^2};$$

$$u_{C2} = \left(-\frac{w_T^2}{2} + b_T w_T\right)\cdot\frac{z-l_1}{h_P^2+(z-l_1)^2}$$

Besides we set $u_{C1} \cong u_{C2} \cong u_T$. The static behaviour of the girder is assumed to follow Engesser-Courbon hypothesis. Finally we denote by $x_{Ci} = \pm b_T$ the mid-width of the girder, by γ_C the specific weight of the stays and by ρ_{CA} the share coefficient relative to the anchorage cables of the resulting force transmitted by the stays to the tower top. The meaning of the other symbols utilized is shown in Figs. 1 and 2.

The system of differential equations governing the vibrations of the bridge has been obtained by the application of Hamilton's principle, that can be expressed as follows:

$$\delta\int_{t_1}^{t_2}(T^* - V_{int}^* - V_g - V_{ext}^*)dt + \int_{t_1}^{t_2}\delta W_D dt = 0$$

where:

- T^* represents the kinetic energy accumulated by the structure when passing from the initial to the final deformed state;
- V_{int}^* represents the strain energy associated with the linear displacement increments;
- V_g represents the geometric strain energy due to initial prestress;

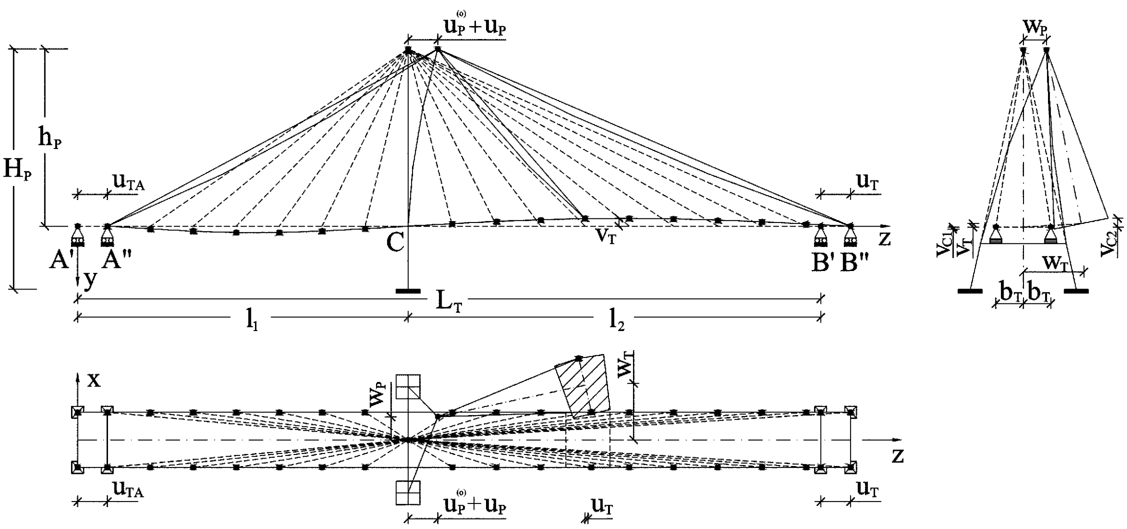

Fig. 2 Deformed configuration

V^*_{ext} represents the potential energy of the additional external forces, which are equal to zero in this case, because the focus is on determining the free vibrations of the bridge;

δW_D is the virtual work made by viscous damping forces, which, as it is known, are non-conservative (Abdel-Ghaffar, Ahmed and Lawrence 1982).

Substituting in the above equation the expressions of the energies reported in Appendix B, we have derived the system of differential equations governing the motion of the structure.

In order to simplify the solution method for the motion equations, the following hypotheses have been made: for the functions

$$C_V = \frac{h_P^2}{h_P^2 + (z - l_1)^2},\ C_U = \frac{z - l_1}{h_P^2 + (z - l_1)^2},\ C_E = \frac{E_c a_c}{(s_C^{(\circ)})^3},$$

variation laws, obtained through a continuous linear regression according to the principle of minimum quadratics, have been adopted; for $s_C^{(\circ)} = \sqrt{(l_1 - z)^2 + b_T^2 + h_p^2}$ the expression obtained through a polynomial regression of the third order, according to the principle of minimum quadratics, has been used.

Functions C_V and C_U appear several times in the functional since they connect respectively the

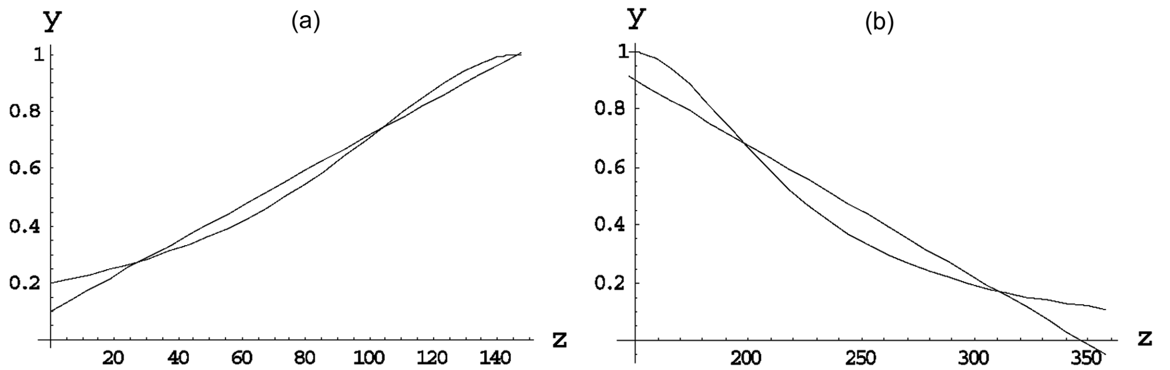

Fig. 3 (a) Law proposed for C_V- left span $C_V = A + Bz = 0.1032 + 0.00614z$, (b) Law proposed for C_V- right span $C_V = C + Dz = 1.5901 - 0.004574z$

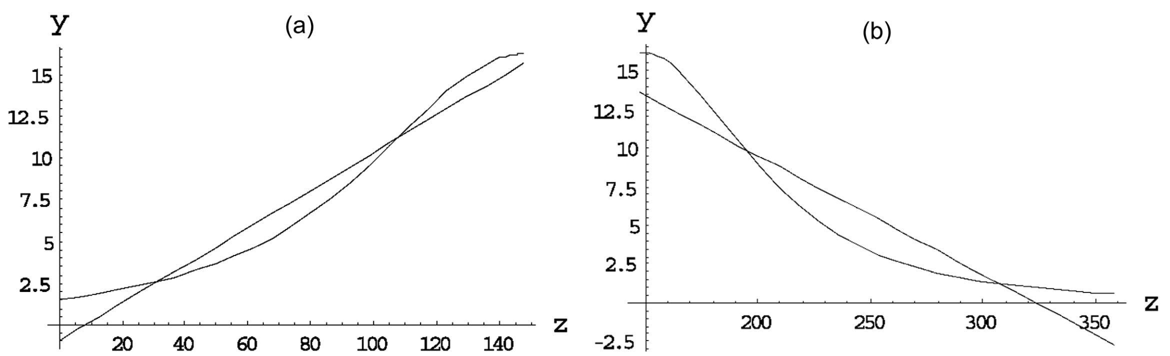

Fig. 4 (a) Law proposed for C_E- left span $C_E = M + Nz = -0.932 + 0.1126z[\mathrm{daN/m^4}]$, (b) Law proposed for C_E- right span $C_E = Q + Rz = 24.96–7.719 \cdot 10^{-2}z[\mathrm{daN/m^4}]$

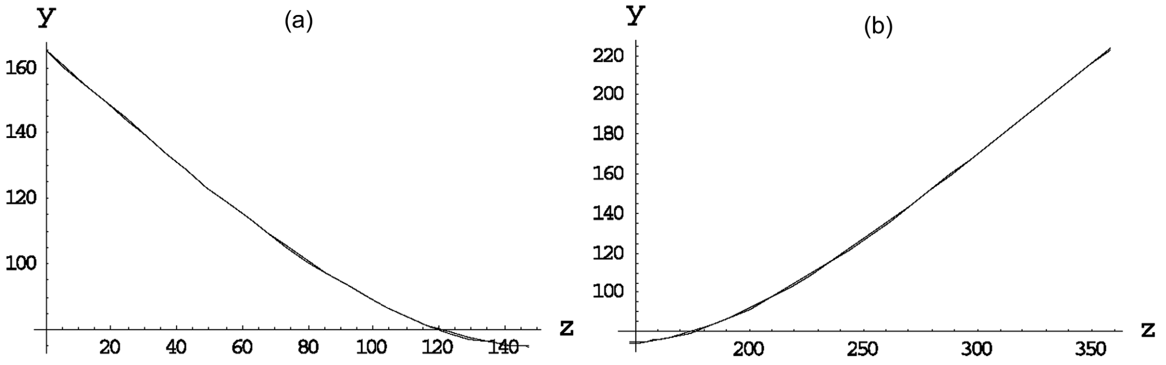

Fig. 5 (a) Law proposed for $s_C^{(°)}$-left span $s_C^{(°)} = F + Gz + Hz^2 + Jz^3 = 165.06 - 0.839z - 8.4 \cdot 10^{-4}z^2 + 1.61 \cdot 10^{-5}z^3$ [m], (b) Law proposed for $s_C^{(°)}$-right span $s_C^{(°)} = K + Sz + Vz^2 + Wz^3 = 195.94 - 1.992z + 9.37 \cdot 10^{-3}z^2 - 1.007 \cdot 10^{-5}z^3$ [m]

Table 1 Data concerning the geometrical and mechanical characteristics of the cable-stayed bridge

$l_1 = 147.42$ [m]	$L_T = 358.02$ [m]	$h_P = 74.2$ [m]	$H_P = 114.6$ [m]	$b_T = 11.73$ [m]
$A_{CA} = 0.034$ [m^2]	$A_C = 0.007$ [m^2]	$a_C = 0.00033$ [m^2/m]	$A_T = 0.74$ [m^2]	$A_P = 18$ [m^2]
$g_T = 193.11$ [kN/m]	$m_T = 19.68$[kN $\cdot$ s^2/m^2]	$\gamma_C = 7.85 \cdot 10^{-5}$ [kN/cm^3]	$J_T = 59.55$ [m^4]	$J_t = 0.18$ [m^4]
$J_P = 112.26$ [m^4]	$E_T = 21000$ [kN/cm^2]	$E_P = 3122$ [kN/cm^2]	$\xi_C = 0.001$	$\xi_T = 0.004$

vertical displacements v_T and v_{Ci} and the longitudinal displacement u_T with the transversal displacement w_T.

Therefore an initial effective simplification of the motion equations has been allowed by their linearization.

Besides, as to function C_E, a mean value of the equivalent tangent Dishinger modulus has been taken into consideration, both for the first and second spans.

Figs 3a-3b, 4a-4b and 5a-5b feature the laws proposed for functions C_V, C_E and $s_C^{(°)}$, concerning a cable-stayed bridge having the geometrical and mechanical characteristics summarized in Table 1 (cf. paragraph 4).

2. Linear solution

The system of differential equations governing the motion of the structure, is reported in Appendix C.

This paper proposes an approximate solution of Eq. (4)', making part of this system. The terms containing the product $w_T \cdot u_P$ and its derivatives, as well as the tower displacement w_P compared to the girder displacement w_T, are neglected: in this way Eq. (4)' becomes indipendent from the other equations of the system. This assumption is allowable since the numerical solutions that have been carried out, have shown that, with the A-shaped tower, the displacements u_P and w_P of its top are negligible with respect to the girder transversal displacement w_T (cf. paragraph 4). Finally the terms in w_T whose order is superior to the first, are neglected too. If the mechanical characteristics of the girder are steady throughout its whole development, the equation describing the damped free transversal vibrations of the girder, putting $E_T J_{xy} = E_T J_t \cdot b_T^2$, assumes the following form:

$$\left(E_T J_T + \frac{E_T J_{xy} C_V^2(z)}{h_P^2}\right)\left(\frac{\partial^4 w_T}{\partial z^4}\right) + \frac{4E_T J_{xy}}{h_P^2} C_V(z)\frac{\partial C_V(z)}{\partial z}\left(\frac{\partial^3 w_T}{\partial z^3}\right) - N_T^{(\circ)}(z)\left(\frac{\partial^2 w_T}{\partial z^2}\right) - \frac{\partial N_T^{(\circ)}(z)}{\partial z}\left(\frac{\partial w_T}{\partial z}\right)$$

$$+\left\{2b_T^2 C_E(z)(1-C_V(z))^2 + \frac{g_T}{h_P}\left[1+\left(\frac{C_V(z)b_T}{h_P}\right)^2\right]\right\} w_T + \left\{\frac{2}{3}\frac{\gamma_C}{g} a_C s_C^{(\circ)}(z)\left[1+\left(\frac{C_V(z)b_T}{h_P}\right)^2\right]\right.$$

$$\left. + m_T\left[1+\left(\frac{C_V(z)b_T^2}{h_P^2}\right)\right]\right\}\left(\frac{\partial^2 w_T}{\partial t^2}\right) + \left\{\frac{2}{3}\xi_C c_{cC} s_C^{(\circ)}(z)\left[1+\left(\frac{C_V(z)b_T}{h_P}\right)^2\right]\right.$$

$$\left. + \xi_T c_{cT}\left[1+\left(\frac{C_V(z)b_T^2}{h_P^2}\right)\right]\right\}\left(\frac{\partial w_T}{\partial t}\right) = 0 \qquad (1)$$

Neglecting the contribution of damping and putting $w_T = w_T^*(z) \cdot e^{i\omega t}$ in steady-state, Eq. (1) becomes:

$$\left(E_T J_T + \frac{E_T J_{xy} C_V^2(z)}{h_P^2}\right)\frac{d^4 w_T^*}{dz^4} + \frac{4E_T J_{xy}}{h_P^2} C_V(z)\frac{dC_V(z)}{dz}\frac{d^3 w_T^*}{dz^3} - N_T^{(\circ)}(z)\frac{d^2 w_T^*}{dz^2} - \frac{dN_T^{(\circ)}(z)}{dz}\frac{dw_T^*}{dz}$$

$$+\left\{2b_T^2 C_E(z)(1-C_V(z))^2 + \left(\frac{g_T}{h_P} - \frac{2}{3}\frac{\gamma_C}{g} a_C \omega^2 s_C^{(\circ)}(z)\right)\left[1+\left(\frac{C_V(z)b_T}{h_P}\right)^2\right] - m_T\omega^2\left[1+\frac{C_V(z)b_T^2}{h_P^2}\right]\right\} w_T^* = 0$$

(2)

Subsequently taking into consideration damping and putting $w_T = w_T^*(z) \cdot \Phi(t)$, in steady-state and in the hypothesis of normal mode of vibration according to which $\omega_{nC} = \omega_{nT} = \omega$ (cf. Appendix B and Monaco and Fiore 2003), Eq. (1) divides into two equations, the first of which coincides with (2) and the second is given by:

$$\Phi(t) + \omega\lambda\Phi(t) + \omega^2\Phi(t) = 0$$

$$\text{with } \lambda = const = \frac{2\cdot\left\{\frac{2}{3}\frac{\gamma_C}{g} a_C \xi_C s_C^{(\circ)}(z)\left[1+\left(\frac{C_V(z)b_T}{h_P}\right)^2\right] + m_T\xi_T\left[1+\left(\frac{C_V(z)b_T^2}{h_P^2}\right)\right]\right\}}{\left\{\frac{2}{3}\frac{\gamma_C}{g} a_C s_C^{(\circ)}(z)\left[1+\left(\frac{C_V(z)b_T}{h_P}\right)^2\right] + m_T\left[1+\left(\frac{C_V(z)b_T^2}{h_P^2}\right)\right]\right\}} \qquad (3)$$

Eq. (3) admits the expression $\Phi(t) = e^{-\frac{\omega\lambda}{2}t} A(\sin(\omega_d t) + \phi)$ as a solution and allows therefore to calculate for each mode of vibration the damped frequency $\omega_d = \omega\sqrt{1-\lambda^2/4}$ and the exponential function $e^{-\frac{\omega\lambda}{2}t}$ governing the variability of motion amplitude in time. Instead (2) is a differential equation with variable coefficients and makes it possibile to obtain the natural (not damped) frequency ω and the shape function $w_T^*(z)$ of the girder axis, for each mode of vibration.

3. Solution of the differential equation

We propose a method for the solution of differential Eq. (2) based on numerical calculus.

Considering initially the boundary value problem for the first span, Eq. (2) can be written, by a synthetic mathematical notation, in the form:

$$A_1(z)w_{1T}^{*\,IV}(z)+B_1(z)w_{1T}^{*\,III}(z)+C_1(z)w_{1T}^{*\,II}(z)+D_1(z)w_{1T}^{*\,I}(z)+E_1(z)w_{1T}^{*} = 0 \text{ with } 0\le z\le l_1 \tag{4}$$

having put:

$$A_1(z) = (a_0+a_1z+a_2z^2) = \left(E_T\,J_T+\frac{A^2E_T\,J_{xy}}{h_P^2}+\frac{2ABE_T\,J_{xy}}{h_P^2}z+\frac{B^2E_T\,J_{xy}}{h_P^2}z^2\right);$$

$$B_1(z) = (b_0+b_1z) = \left(\frac{4ABE_T\,J_{xy}}{h_P^2}+\frac{4B^2E_T\,J_{xy}}{h_P^2}z\right);$$

$$C_1(z) = (c_0+c_1z+c_2z^2) = \left(\rho_{CA}\frac{g_T}{2h_P}((L_T-l_1)^2-l_1^2)+\frac{g_Tl_1}{h_P}z-\frac{g_T}{2h_P}z^2\right);$$

$$D_1(z) = (d_0+d_1z) = \left(\frac{g_Tl_1}{h_P}-\frac{g_T}{h_P}z\right);$$

$$\begin{aligned}E_1(z) &= (e_0+e_1z+e_2z^2+e_3z^3+e_4z^4+e_5z^5) = \left\{2Mb_T^2(1-A)^2+\left(\frac{g_T}{h_P}-\frac{2}{3}\frac{\gamma_C}{g}a_C\omega^2F\right)\left[1+\left(\frac{Ab_T}{h_P}\right)^2\right]\right.\\
&-m_T\omega^2\left(1+\frac{Ab_T^2}{h_P^2}\right)+\left[4BMb_T^2(A-1)+2Nb_T^2(1-A)^2+2\frac{g_T}{h_P^3}ABb_T^2-\frac{2}{3}\frac{\gamma_C}{g}a_C\omega^2\left(G+\frac{2ABFb_T^2}{h_P^2}+\frac{A^2Gb_T^2}{h_P^2}\right)\right.\\
&\left.-\frac{m_T\omega^2Bb_T^2}{h_P^2}\right]z+\left[2B^2Mb_T^2+4BNb_T^2(A-1)+\frac{g_T}{h_P^3}B^2b_T^2-\frac{2}{3}\frac{\gamma_C}{g}a_C\omega^2\left(H+\frac{B^2Fb_T^2}{h_P^2}+\frac{2ABGb_T^2}{h_P^2}+\frac{A^2Hb_T^2}{h_P^2}\right)\right]z^2\\
&+\left[2B^2Nb_T^2-\frac{2}{3}\frac{\gamma_C}{g}a_C\omega^2\left(J+\frac{B^2Gb_T^2}{h_P^2}+\frac{2ABHb_T^2}{h_P^2}+\frac{A^2Jb_T^2}{h_P^2}\right)\right]z^3-\left[\frac{2}{3}\frac{\gamma_C}{g}a_C\omega^2\left(\frac{B^2Hb_T^2}{h_P^2}+\frac{2ABJb_T^2}{h_P^2}\right)\right]z^4\\
&\left.-\left[\frac{2}{3}\frac{\gamma_C}{g}a_C\omega^2\frac{B^2Jb_T^2}{h_P^2}\right]z^5\right\}.\end{aligned}$$

To transform the boundary value problem of the fourth order into a problem of the second order, it is necessary to replace the expression $y_1(z) = w_{1T}^{*\,II}(z)$ in Eq. (4), getting the following system:

$$\begin{cases}A_1(z)y_1^{II}(z)+B_1(z)y_1^{I}(z)+C_1(z)y_1(z)+D_1(z)w_{1T}^{*\,I}(z)+E_1(z)w_{1T}^{*}(z) = 0\\ y_1(z)-w_{1T}^{*\,II}(z) = 0\end{cases} \tag{5}$$

The latter is rewritten in matrix form:

$$\alpha_1(z)h^{II}(z)+\alpha_2(z)h^{I}(z)+\alpha_3(z)h(z) = 0 \qquad 0\le z\le l_1 \tag{6}$$

with:

$$h(z) = \begin{pmatrix} y_1(z) \\ w^*_{1T}(z) \end{pmatrix}; \quad \alpha_1(z) = \begin{pmatrix} A_1(z) & 0 \\ 0 & -1 \end{pmatrix};$$

$$\alpha_2(z) = \begin{pmatrix} B_1(z) & D_1(z) \\ 0 & 0 \end{pmatrix}; \quad \alpha_3(z) = \begin{pmatrix} C_1(z) & E_1(z) \\ 1 & 0 \end{pmatrix}$$

The boundary value problem for the second span is obtained in a perfectly analogous way:

$$\beta_1(z)h^{II}(z) + \beta_2(z)h^{I}(z) + \beta_3(z)h(z) = 0 \qquad l_1 \le z \le L_T \tag{7}$$

with:

$$h(z) = \begin{pmatrix} y_2(z) \\ w^*_{2T}(z) \end{pmatrix}; \quad \beta_1(z) = \begin{pmatrix} A_2(z) & 0 \\ 0 & -1 \end{pmatrix};$$

$$\beta_2(z) = \begin{pmatrix} B_2(z) & D_2(z) \\ 0 & 0 \end{pmatrix}; \quad \beta_3(z) = \begin{pmatrix} C_2(z) & E_2(z) \\ 1 & 0 \end{pmatrix};$$

$$A_2(z) = (a_0 + a_1 z + a_2 z^2) = \left(E_T J_T + \frac{C^2 E_T J_{xy}}{h_P^2} + \frac{2CDE_T J_{xy}}{h_P^2} z + \frac{D^2 E_T J_{xy}}{h_P^2} z^2 \right);$$

$$B_2(z) = (b_0 + b_1 z) = \left(\frac{4CDE_T J_{xy}}{h_P^2} + \frac{4D^2 E_T J_{xy}}{h_P^2} z \right);$$

$$C_2(z) = (c_0 + c_1 z + c_2 z^2) = \left(\frac{g_T}{2h_P}((L_T - l_1)^2 - l_1^2) + \frac{g_T l_1}{h_P} z - \frac{g_T}{2h_P} z^2 \right);$$

$$D_2(z) = (d_0 + d_1 z) = \left(\frac{g_T l_1}{h_P} - \frac{g_T}{h_P} z \right); \quad E_2(z) = (e_0 + e_1 z + e_2 z^2 + e_3 z^3 + e_4 z^4 + e_5 z^5)$$

$$= \left\{ 2Qb_T^2(1-C)^2 + \left(\frac{g_T}{h_P} - \frac{2}{3}\frac{\gamma_C}{g} a_C \omega^2 K \right)\left[1 + \left(\frac{Cb_T}{h_P} \right)^2 \right] - m_T \omega^2 \left(1 + \frac{Cb_T^2}{h_P^2} \right) \right.$$

$$+ \left[4DQb_T^2(C-1) + 2Rb_T^2(1-C)^2 + 2\frac{g_T}{h_P^3} CDb_T^2 - \frac{2}{3}\frac{\gamma_C}{g} a_C \omega^2 \left(S + \frac{2CDKb_T^2}{h_P^2} + \frac{C^2 S b_T^2}{h_P^2} \right) - \frac{m_T \omega^2 D b_T^2}{h_P^2} \right] z$$

$$+ \left[2D^2 Q b_T^2 + 4DRb_T^2(C-1) + \frac{g_T}{h_P^3} D^2 b_T^2 - \frac{2}{3}\frac{\gamma_C}{g} a_C \omega^2 \left(V + \frac{D^2 K b_T^2}{h_P^2} + \frac{2CDSb_T^2}{h_P^2} + \frac{C^2 V b_T^2}{h_P^2} \right) \right] z^2$$

$$+ \left[2D^2 R b_T^2 - \frac{2}{3}\frac{\gamma_C}{g} a_C \omega^2 \left(W + \frac{D^2 S b_T^2}{h_P^2} + \frac{2CDVb_T^2}{h_P^2} + \frac{C^2 W b_T^2}{h_P^2} \right) \right] z^3 - \left[\frac{2}{3}\frac{\gamma_C}{g} a_C \omega^2 \left(\frac{D^2 V b_T^2}{h_P^2} + \frac{2CDWb_T^2}{h_P^2} \right) \right] z^4$$

$$\left. - \left[\frac{2}{3}\frac{\gamma_C}{g} a_C \omega^2 \frac{D^2 W b_T^2}{h_P^2} \right] z^5 \right\}.$$

To solve the equation under examination from a numerical view-point, it is necessary to consider the boundary value problems related to the two spans as just one discrete problem defined over the whole length of the bridge. For this purpose a decomposition of interval $[0, l_1]$ is initially carried out, that is to say, putting $K = l_1/N$, with N as a whole number, the following points are defined:

$$0 = z_0 < z_1 < z_2 < \ldots\ldots < z_{N-1} < z_N = l_1 \text{ with } z_j = jK, \quad j = 0, \ldots, N.$$

If interval $[l_1, L_T]$ is then divided using the same amplitude K chosen for the first span, indicating the lower whole part of the real number K by $\lfloor K \rfloor$, the following points are obtained:

$$l_1 = z_N < z_{N+1} < z_{N+2\ldots} < z_{N+M} = L_T \text{ with } M = \frac{L_T - l_1}{\lfloor K \rfloor}.$$

Subsequently the first and second derivatives of the unknown fuctions are approximated using finite difference formulas based on Taylor expansions. Denoting the approximation of the function in point $z = z_j$ by h_j, that is setting $h_j \cong h(z_j)$ with $j = 0, \cdots, N+M$, the first and second derivatives of h_j can be witten:

$$h^I(z_j) \cong \frac{h_{j+1} - h_{j-1}}{2K}; \quad h^{II}(z_j) \cong \frac{h_{j-1} - 2h_j + h_{j+1}}{K^2} \tag{8}$$

On the basis of the above assumptions, Eqs. (6) and (7) become:

$$\alpha_1(z_j)h^{II}(z_j) + \alpha_2(z_j)h^I(z_j) + \alpha_3(z_j)h(z_j) = 0 \qquad \text{with } 1 \le j \le N-1 \tag{9}$$

$$\beta_1(z_{j+N})h^{II}(z_{j+N}) + \beta_2(z_{j+N})h^I(z_{j+N}) + \alpha_3(z_{j+N})h(z_{j+N}) = 0 \quad \text{with } 1 \le j \le M-1 \tag{10}$$

and, replacing the first and second derivatives with expressions (8), it follows that:

$$[2\alpha_1(z_j) - K\alpha_2(z_j)]h_{j-1} + [2K^2\alpha_3(z_j) - 4\alpha_1(z_j)]h_j + [2\alpha_1(z_j) + K\alpha_2(z_j)]h_{j+1} = 0$$

with $1 \le j \le N-1$ (11)

$$[2\beta_1(z_{N+j}) - K\beta_2(z_{N+j})]h_{N+j-1} + [2K^2\beta_3(z_{N+j}) - 4\beta_1(z_{N+j})]h_{N+j}$$
$$+ [2\beta_1(z_{N+j}) + K\beta_2(z_{N+j})]h_{N+j+1} = 0 \qquad \text{with } 1 \le j \le M-1 \tag{12}$$

A system of $N + M - 2$ vectorial equations is formed by Eqs. (11) and (12) altoghether.

Finally, putting $\bar{h} = (h_1, h_2, \ldots, h_{N-1}, h_{N+1}, \ldots, h_{N+M-1})^T$ and defining the following 2×2 matrices:

$$\overline{A}_j = 2\alpha_1(z_j) - K\alpha_2(z_j) \qquad \text{with } j = 1, \ldots, N-1;$$

$$\overline{B}_j = 2K^2\alpha_3(z_j) - 4\alpha_1(z_j) \qquad \text{with } j = 1, \ldots, N-1;$$

$$\overline{C}_j = 2\alpha_1(z_j) + K\alpha_2(z_j) \qquad \text{with } j = 1, \ldots, N-2;$$

$$\overline{D}_j = 2\beta_1(z_{N+j}) - K\beta_2(z_{N+j}) \qquad \text{with } j = 1, \ldots, M-1$$

$$\overline{E}_j = 2K^2\beta_3(z_{N+j}) - 4\beta_1(z_{N+j}) \qquad \text{with } j = 1, \ldots, M-1;$$

$$\overline{F}_j = 2\beta_1(z_{N+j}) + K\beta_2(z_{N+j}) \qquad \text{with } j = 1, \ldots, M-2 \quad (13)$$

a block tridiagonal linear system is obtained. The boundary conditions, related to the structure under examination, to be satisfied at the ends of each span, are:

$$w^*_{1T}(0) = 0; \quad w^*_{1T}(l_1) = w^*_{2T}(l_1) = 0; \quad w^*_{2T}(L_T) = 0; \quad \frac{\partial^2 w^*_{1T}}{\partial z^2}(0) = 0;$$

$$\frac{\partial^2 w^*_{2T}}{\partial z^2}(L_T) = 0; \quad \frac{\partial w^*_{1T}}{\partial z}(l_1) = \frac{\partial w^*_{2T}}{\partial z}(l_1); \quad \frac{\partial^2 w^*_{1T}}{\partial z^2}(l_1) = \frac{\partial^2 w^*_{2T}}{\partial z^2}(l_1) \quad (14)$$

From the conditions related to points z=0 and z=L_T it follows that:

$$h_0 = h(z_0) = 0; \quad h_{N+M} = h(L_T) = 0 \quad (15)$$

On point of ascissa $z = l_1$ instead, indicating by h^1_N and h^2_N respectively the first and second components of vector h_N, according to conditions (14), we can set:

$$\frac{\partial^2 w^*_{1T}}{\partial z^2}(l_1) = \frac{\partial^2 w^*_{2T}}{\partial z^2}(l_1) = h^1_N; \quad w^*_{1T}(l_1) = w^*_{2T}(l_1) = h^2_N = 0 \quad (16)$$

Consequently the vectorial equations regarding abscissa points $z = z_{N\text{-}1}$ and $z = z_{N+1}$ can be written as follows:

$$\overline{A}_{N-1}h_{N-2} + \overline{B}_{N-1}h_{N-1} + \overline{C}_{N-1}h^1_N = 0; \quad \overline{D}_1 h^1_N + \overline{E}_1 h_{N+1} + \overline{F}_1 h_{N+2} = 0 \quad (17)$$

with:

$$\overline{C}_{N-1} = \begin{pmatrix} 2A_1(z_{N-1}) + KB_1(z_{N-1}) \\ 0 \end{pmatrix}; \quad \overline{D}_1 = \begin{pmatrix} 2A_2(z_{N+1}) - KB_2(z_{N+1}) \\ 0 \end{pmatrix}$$

In addition, having adopted the same interval amplitude both in the first and the second spans and symmetric approximations, the continuity of the first, third and fourth derivatives on point $z = l_1$ results automatically satisfied. Conversely recalling the meaning of vector $h(z_N)$, the third and fourth derivatives of the unknown function on the abscissa point $z = l_1$ have to be differentiated, namely also conditions (18) must be satisfied:

$$\frac{\partial^3 w^*_{1T}}{\partial z^3}(l_1) \neq \frac{\partial^3 w^*_{2T}}{\partial z^3}(l_1); \quad \frac{\partial^4 w^*_{1T}}{\partial z^4}(l_1) \neq \frac{\partial^4 w^*_{2T}}{\partial z^4}(l_1) \quad (18)$$

For this purpose, paying attention to the equations concerning the two spans on $z = l_1$, expressed respectively by:

$$A_i(l_1)\frac{d^4 w^*_{iT}}{dz^4}(l_1) + B_i(l_1)\frac{d^3 w^*_{iT}}{dz^3}(l_1) + C_i(l_1)\frac{d^2 w^*_{iT}}{dz^2}(l_1) + D_i(l_1)\frac{dw^*_{iT}}{dz}(l_1) + E_i(l_1, \omega)w^*_{iT}(l_1) = 0$$

with $i = 1, 2$ (19)

and recalling the boundary conditions, 4 unknown values remain to be approximated, that is the third and fourth derivatives to the right and the left of the point. For the third derivatives it is possible to apply the backward and the forward difference formulas respectively for the spans on the left and on the right of $z = l_1$:

$$\frac{d^3 \overset{*}{w}_{1T}}{dz^3}(l_1) \cong \frac{\overset{*}{w}_{1T}^{II}(l_1) - \overset{*}{w}_{1T}^{II}(l_1 - K)}{K} = \frac{h_N^1 - h_{N-1}^1}{K};$$

$$\frac{d^3 \overset{*}{w}_{2T}}{dz^3}(l_1) \cong \frac{\overset{*}{w}_{2T}^{II}(l_1 + K) - \overset{*}{w}_{2T}^{II}(l_1)}{K} = \frac{h_{N+1}^1 - h_N^1}{K} \tag{20}$$

Instead as to the approximation of the fourth derivatives, it is rather complex since the known formulas cannot be applied to noncontinuous functions. Therefore one can think of proceeding in the following way: two new unknown values are introduced into the system,

$$x = \frac{d^4 \overset{*}{w}_{1T}}{dz^4}(l_1) \text{ and } y = \frac{d^4 \overset{*}{w}_{2T}}{dz^4}(l_1),$$

which are inserted in the equations of the two spans. But doing so an equation is absent because the unknown values to be determined have become three (h_N^1, x, and y). In order to find the latter information the following procedure can be used: the first derivatives are approximated applying the forward and backward difference formulas and then the continuity condition is explicity added. The approximate equation for the first span becomes:

$$KA_1(l_1)x + (B_1(l_1) + KC_1(l_1))h_N^1 - B_1(l_1)h_{N-1}^1 - D_1(l_1)h_{N-1}^2 = 0 \tag{21}$$

Analogously the approximate equation for the second span can be written:

$$KA_2(l_1)y + (KC_2(l_1) - B_2(l_1))h_N^1 + B_2(l_1)h_{N+1}^1 + D_2(l_1)h_{N+1}^2 = 0 \tag{22}$$

The continuity equation of the first derivatives instead is given by:

$$\frac{h_N^2 - h_{N-1}^2}{K} = \frac{h_{N+1}^2 - h_N^2}{K} \Rightarrow -h_{N-1}^2 = h_{N+1}^2 \Rightarrow h_{N-1}^2 + h_{N+1}^2 = 0 \tag{23}$$

In conclusion, after defining the following order of the unknown values in proximity to point $z = l_1$:

$$h_{N-1} = \begin{bmatrix} h_{N-1}^1 \\ h_{N-1}^2 \end{bmatrix}; \quad h_N = \begin{bmatrix} h_N^1 \\ x \\ y \end{bmatrix}; \quad h_{N+1} = \begin{bmatrix} h_{N+1}^1 \\ h_{N+1}^2 \end{bmatrix};$$

we obtain the system:

$$\begin{cases} -B_1(l_1)h_{N-1}^1 - D_1(l_1)h_{N-1}^2 + (B_1(l_1) + KC_1(l_1))h_N^1 + KA_1(l_1)x = 0; \\ (KC_2(l_1) - B_2(l_1))h_N^1 + KA_2(l_1)y + B_2(l_1)h_{N+1}^1 + D_2(l_1)h_{N+1}^2 = 0; \\ h_{N-1}^2 + h_{N+1}^2 = 0; \end{cases}$$

which in matrix form is represented by:

$$\overline{A}_N h_{N-1} + \overline{B}_N h_N + \overline{C}_N h_{N+1} = 0 \tag{24}$$

with:

$$\overline{A}_N = \begin{pmatrix} -B_1(l_1) & -D_1(l_1) \\ 0 & 0 \\ 0 & 1 \end{pmatrix}; \quad \overline{B}_N = \begin{pmatrix} B_1(l_1)+KC_1(l_1) & KA_1(l_1) & 0 \\ KC_2(l_1)-B_2(l_1) & 0 & KA_2(l_1) \\ 0 & 0 & 0 \end{pmatrix}; \quad \overline{C}_N = \begin{pmatrix} 0 & 0 \\ B_2(l_1) & D_2(l_1) \\ 0 & 1 \end{pmatrix}$$

On the basis of these considerations, the global system to be solved is equivalent to the product of a block tridiagonal matrix, indicated by A, for vector $\overline{h}$:

$$[A]\cdot\overline{h} = \begin{bmatrix} \overline{B}_1 & \overline{C}_1 & & & & & & & & \\ \overline{A}_2 & \overline{B}_2 & \overline{C}_2 & & & & & & & \\ & \overline{A}_3 & \overline{B}_3 & \overline{C}_3 & & & & & & \\ & & \dots & \dots & \dots & & & & & \\ & & & \overline{A}_{N-1} & \overline{B}_{N-1} & \overline{C}_{N-1} & & & & \\ & & & & \overline{A}_N & \overline{B}_N & \overline{C}_N & & & \\ & & & & & \overline{A}_{N+1} & \overline{B}_{N+1} & \overline{C}_{N+1} & & \\ & & & & & & \dots & \dots & \dots & \\ & & & & & & & \overline{D}_{M-3} & \overline{E}_{M-3} & \overline{F}_{M-3} \\ & & & & & & & & \overline{D}_{M-2} & \overline{E}_{M-2} \; \overline{F}_{M-2} \\ & & & & & & & & & \overline{D}_{M-1} \; \overline{E}_{M-1} \end{bmatrix} \cdot \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \dots \\ h_{N-1} \\ h_N \\ h_{N+1} \\ \dots \\ h_{N+M-3} \\ h_{N+M-2} \\ h_{N+M-1} \end{bmatrix} \qquad (25)$$

The last step is to study for which values of parameter ω the problem allows a nontrivial solution, that is to say coefficient matrix A results singular. In order to do that, it is useful to recall the notions of eigenvector $\kappa_i \in C^n$ and eigenvalue $\lambda_i \in C$ of an $A \in C^{n\times n}$ matrix: $A\kappa_i = \lambda_i\kappa_i$. Besides it is necessary to apply the relation connecting the determinant of an $A \in C^{n\times n}$ matrix to its eigenvalues λ_i:

$$\det A = \prod_{i=1}^{n}\lambda_i \qquad (26)$$

By program *"main1"*, reported in Appendix D, matrix $A(\omega)$ has been rebuilt.

This program utilizes automatically matrices α_1, α_2, α_3, β_1, β_2, β_3 and elements A_1, B_1, C_1, D_1, E_1, A_2, B_2, C_2, D_2, E_2, previously defined, after they have been created in M-files and memorized in the work directory. Program *"main1"* has been structured in such a way to provide as *output*, for each $n \in \Re$ value of ω inserted in *input*, the lesser eigenvalue ($\lambda_{\min}$), in absolute value, within the whole set of eigenvalues λ_i corresponding to matrix $A(\omega = n)$ and the graph of the girder transversal displacements normalized to the unit, which, apart from the constrained points, coincide with the second component of the eigenvector related to $\lambda_{\min}(\omega = n)$. Specifically this program allows to plot an approximate graph of function $w_T^*(z)$, utilizing a spline function to obtain a continuous solution in interval $[0, L_T]$, starting from the discrete points where the solution has been approximated by the numerical method. The number of points chosen to calculate the value of the continuous approximation is 750, that is more than twice the points chosen for the discretization.

Executing the program for a sufficient number of values of ω, it is possible to diagram the obtained eigenvalues $\lambda_{\min}$ in function of the values of ω themselves: in the light of Eq. (26), the abscissas of minimum points of the resulting curve are just the values of ω making the determinant of matrix A equal to zero and then the natural frequencies that have been looked for.

4. Numerical application

The proposed solution method has been utilized to determine the frequencies of the cable-stayed bridge having the characteristics reported in Table 1 and in Fig. 6.

The results of the calculus procedure are synthesized in Fig. 7, where the first four vibration frequencies of the bridge (the abscissas of the null points of the graph) are underlined.

Finally we have performed the dynamic analysis of the cable-stayed bridge under examination by a finite elements calculation program. Finite elements code SAP2000 has been used. The initial stage consisted in building a model as faithful as possible to the structure under examination, whose geometrical and mechanical features are summarized in Table 1 and in Fig. 6. The geometric realization of the model has been obtained, properly placing, with respect for the real configuration of the structure, a series of linear elements (frame elements) and of plane elements (shell elements), interconnected by joints. In particular frame elements have been used both for the pylon and the stays, shell elements for the girder, with a total of 568 frame elements, 2035 shell elements, 1600 joints and 9528 degrees of freedom. Each frame or shell element is associated to a section and a material, coherently to the data reported in Table 1. Precisely only three materials have been employed, the concrete which the pylon is made of, and two types of steel, one for the stays and the other for the girder. Program SAP2000 provides the natural frequencies of the structure related to wholly general modes of vibration. Therefore, from the first 30 modes of vibration of the bridge, we

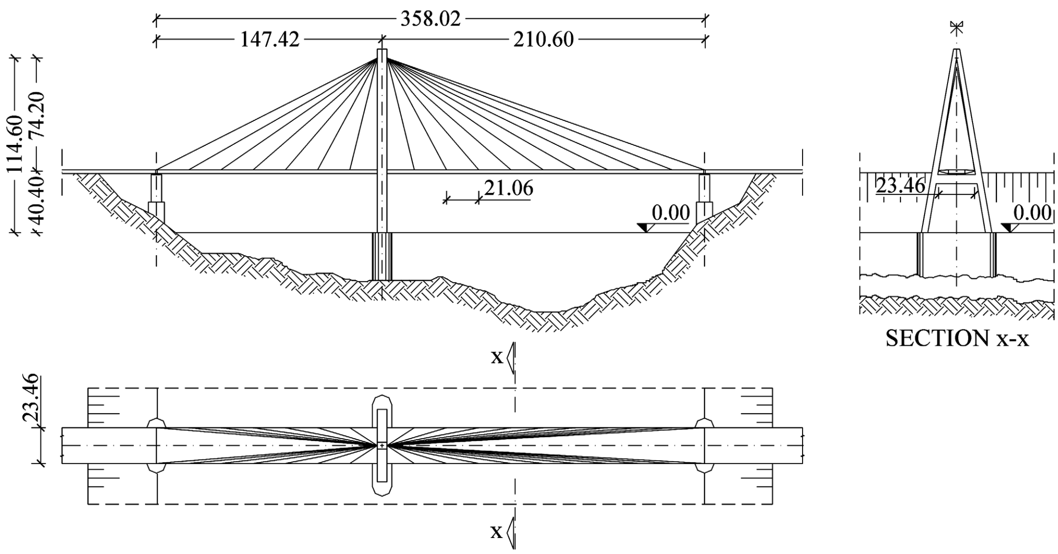

Fig. 6 Geometrical characteristics of the cable-stayed bridge

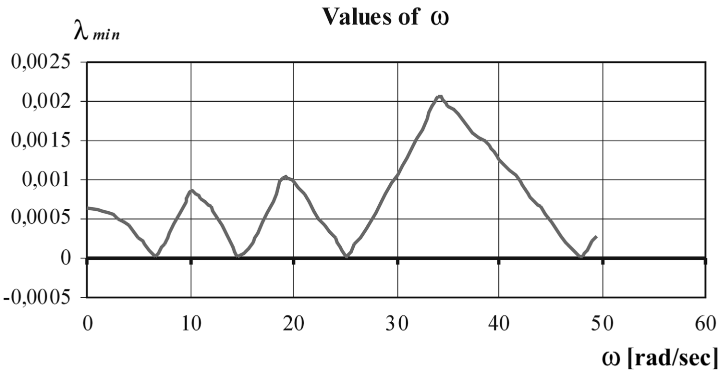

Fig. 7 Graph of the natural frequencies

have extrapolated only the vibrations having the transversal displacement component of the girder predominant in comparison with the other motion components. In particular the transversal modes of vibration have resulted to be the 4th, the 12th, the 18th and the 30th ones. From now onwards, these modes of vibration will be referred to as respectively the 1st, the 2nd, the 3rd and the 4th transversal modes of vibration of the bridge girder. Obviously the first 30 modes of vibration also include the vertical, longitudinal and torsional modes of vibration and the modes with coupled motion of the girder and the tower (Abdel-Ghaffar 1991). Just one of these coupled motions concerns the transversal oscillations of the girder, which in this case are indeed decidedly negligible in comparison with the vibrations of the tower: that is the reason why these motions lie outside the research carried out.

Observing Table 2 you can note that the values of natural frequencies ω_i obtained by the numerical method are quite comparable to the values determined by the finite elements procedure.

Moreover, from a comparison between the displacements of the tower top and those of the girder, derived from program SAP2000, it results that the hypothesis of neglecting the displacements of the tower top compared to the transversal one of the girder can be considered acceptable. In fact Table 3 shows, for each of the four transversal modes of vibration, respectively the values of the displacements of the tower top and the value of the maximum transversal displacement of the girder.

Table 2 Values of the natural frequencies

Numerical method	ω_1 = 6.54 rad/sec	ω_2 = 14.65 rad/sec	ω_3 = 25.2 rad/sec	ω_4 = 47.75 rad/sec
Finite elements	ω_1 = 7.78 rad/sec	ω_2 = 16.35 rad/sec	ω_3 = 25.67 rad/sec	ω_4 = 44.12 rad/sec

Table 3 Values of the displacements

Mode	ω [rad/sec]	Displacements of the tower top			Maximum displacement of the girder in direction x [mm.]
		Displacement in direction x [mm.]	Displacement in direction y [mm.]	Displacement in direction z [mm.]	
1	7.78	0	$3.408*10^{-5}$	0	0.02943
2	16.35	0	$1.776*10^{-4}$	0	0.02665
3	25.67	0	$4.95*10^{-4}$	$1.404*10^{-6}$	0.02493
4	44.12	$-2.034*10^{-6}$	$-2.371*10^{-5}$	0	0.02784

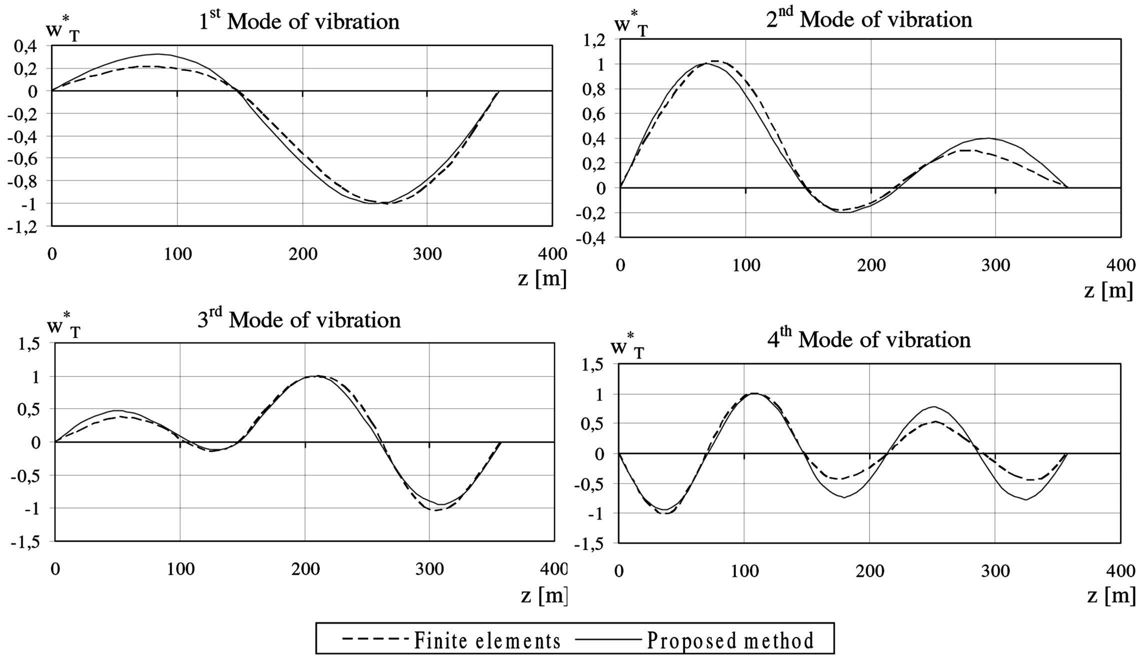

Fig. 8 Modes of vibration

Finally Fig. 8 shows the shape-functions of the girder axis normalized to the unit, related to the first four transversal modes of vibration, obtained by the two above mentioned methods.

5. Conclusions

After knowing the system of differential equations governing the motion of a self-anchored cable-stayed bridge with a curtain suspension, obtained by a study to the continuum based on generalized Hamilton's principle in a nonlinear field, neglecting nonlinear terms and some quantities affecting the solution in a non-remarkable way, the equation of the transversal oscillatory motion of the girder of the bridge has been derived.

This equation, in the hypothesis of normal mode of vibration and in steady-state, has been divided into two equations: the first is a fourth order differential equation with variable coefficients, represents the equation concerning the free transversal oscillations of the girder and makes it possibile to obtain, for each mode of vibration, the natural (not damped) frequency and the shape function of the girder axis; the second one is a second order differential equation, reflects the damping of the system and allows to calculate, for each value of the natural frequency, the corresponding damped frequency and the exponential function governing the variability of motion amplitude in time. The equation describing the free (not damped) transversal motion of the bridge girder has been solved by a numerical method. Initially the boundary value problem of the fourth order has been transformed into a problem of the second order. Subsequently, after dividing the first and second spans respectively into N and M intervals having the same amplitude, for each interval the motion equation has been rewritten and the first and second derivatives of the unknown functions have been approximated using finite difference formulas based on Taylor expansions. Then the boundary conditions have been imposed at the ends of each span and a vectorial system, characterized by a block tridiagonal coefficient matrix, has consequently been obtained. Finally a study has been carried out to determine for which values of natural frequencies the problem allows a nontrivial solution, that is to say the coefficient matrix results singular.

In order to do this, program "*main1*", based on the relation connecting the determinant of a matrix to its eigenvalues, has been elaborated. In particular program *"main1"* has been structured in such a way to provide as *output*, for each value of natural frequency inserted in *input*, the lesser eigenvalue, in absolute value, within the whole set of eigenvalues corresponding to the coefficient matrix calculated for that specific value of the natural frequency and the graph of the girder transversal displacements normalized to the unit.

So it has been possible to diagram the *minimum eigenvalues* so obtained in function of the values of the natural fequencies themselves, putting the values of natural frequencies in abscissa and the values of *minimum eigenvalues* in ordinate: the abscissas of minimum points of the resulting curve have provided just the natural frequencies that have been looked for.

Finally the dynamic analysis of the bridge has been performed by using a finite elements calculation program. Finite elements code SAP2000 has been used. This program provides the natural frequencies of the structure related to wholly general modes of vibration, thereby only the vibrations having the transversal displacement component of the girder predominant in comparison with the other motion components have been extrapolated from the first 30 modes of vibration including the transversal, longitudinal, vertical and torsional motions of the bridge. The coincidence between the results obtained by the two different methods attests the validity of the treatment we have conducted. The numerical method that has been presented in this paper, has led to appreciable results, proving to be better than a finite elements calculation program, both for the application speed of the procedure and its suitability to the several typoligies of cable-stayed bridges. Moreover the numerical method we have proposed, allows to take into account some specificities of the structure being examined, that can hardly be inserted into a finite elements calculation program.

References

Abdel-Ghaffar, A. M. (1976), *Dynamic Analyses of Suspension Bridge Structures*, California Institute of Tecnology-Earthquake Engineering Research Laboratory.

Abdel-Ghaffar, A. M. (1991), "Cable-stayed bridges under seismic action", *Cable-stayed Bridges-Recent Developments and their Future, Proceedings of the Seminar, Yokohama, Japan, 10-11 December 1991*, edited by Ito, M., et al., Elsevier Science Publishers B.V., Amsterdam.

Abdel-Ghaffar, Ahmed, M. and R., Lawrence, I. (1982), "Multiple-support excitations of suspension bridges", *J. Eng. Mech.*, ASCE, **108**(EM2).

Albi Marini, A., Augenti, N., Raithel, A. (1979), *Introduzione alla Dinamica dei ponti strallati*, Atti dell'Istituto di Costruzioni di Ponti, Facoltà di Ingegneria della Università di Napoli, Italy.

Bartoli, G., Ricciardelli, F., Sepe, V. (2004), *WINDERFUL Wind and INfrastructures: Dominating Eolian Risk For Utilities and Lifelines*, Firenze University Press, Firenze (Italy).

Bartoli, G., Cluni, F., Gusella, V., Procino, L. (2004), "Dinamica del cavo sotto l'azione del vento: analisi sperimentale in galleria del vento", *Proceedings of IN-VENTO-2004 8° Convegno Nazionale di Ingegneria del Vento*, Reggio Calabria (Italy), Giugno.

De Miranda, F., Grimaldi, A., Maceri, F., Como, M. (1979), *Basic Problems in Long Span Cable Stayed Bridges*,

University of Calabria, Italy.
Gimsing, N. J. Technical University of Denmark (1983), *Clable Supported Bridges*, John Wiley & Sons, New York.
Greenspan, D. and Casulli, V. (1988), *Numerical Analysis for applied mathematics, Science, and Engineering*, Addison-Wesley Publishing Company, New York.
Ito, M., Fujino, Y., Miyata, T. and Narita, N. (1991), *Cable-stayed Bridges-Recent Developments and Their Future*, Elsevier Science Publishers B.V., Amsterdam.
Leonhardt, F. and Zellner, W. (1970), *Cable-Stayed Bridges: Report on Latest Developments*, Canadian Structural Engineering Conference, Toronto.
Leonhardt, F. and Zellner, W. (1991), "Past, present and future of cable-stayed bridges", *Cable-stayed Bridges-Recent Developments and their Future, Proceedings of the Seminar, Yokohama, Japan, 10-11 December 1991*, edited by Ito, M., et al., Elsevier Science Publishers B.V., Amsterdam.
Monaco, P. and Fiore, A. (2003), "Free transversal vibrations in self-anchored cable-stayed bridges with a curtain suspension", *Proceedings of the Fifth International Symposium on Cable Dynamics*, Santa Margherita Ligure (Italy), September.
Monaco, P. (1995), *I ponti strallati. 1^a e 2^a parte*, Rassegna tecnica pugliese 1/2.1995, Ass. Reg. Ing. e Arch. di Puglia, Italy.
Salvatori, L., Spinelli, P. (2004), "Influenza delle non-linearità strutturali sulla risposta dei ponti sospesi all'azione eolica: simulazioni numeriche su strutture semplificate", *Proceedings of IN-VENTO-2004 8° Convegno Nazionale di Ingegneria del Vento*, Reggio Calabria (Italy), Giugno.
Thomson, W. T. (1965), *Theory of Vibration and applications*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey.
Warburton (1976), *The Dynamical Behaviour of Structures*, Pergamon Press Ltd, Headington Hill Hall, Oxford OX3 0BW, England.
Wells, D. A. (1984), *Collana SCHAUM. Teoria e problemi di Dinamica Lagrangiana*, Etas Libri, Milano (Italy).

Appendix B: Potential energy, kinetic energy and virtual work of the viscous damping forces

Potential energy of the structure

The potential energy of the whole structure is given by the sum of the contributions relative to the stays, the girder and the pylon: $V^*_{int} + V_g = (V^*_{int} + V_g)_{CA} + (V^*_{int} + V_g)_C + (V^*_{int} + V_g)_T + (V^*_{int})_P$.

<u>Potential energy of the anchorage stays:</u>

$$(V^*_{int} + V_g)_{CA} = \sum_{i=1,2}\left\{\frac{E_{CA}A_{CA}}{8(s^{(\circ)}_{CA})^3}[(u_{TA} - u_P)^2 + (w_P)^2 - 2x_{Ci}(w_p) - 2l_1(u_{TA} - u_p)]^2 + \frac{S^{(\circ)}_{CA}}{2s^{(\circ)}_{CA}}[(u_{TA} - u_P)^2 + (w_P)^2]\right\}$$

with: $$S^{(\circ)}_{CA} = \frac{\rho_{CA}s^{(\circ)}_{CA}g_T}{2l_1}\frac{((L_T - l_1)^2 - l_1^2)}{2h_P};\quad \rho_{CA} = \frac{2E_{CA}A_{CA}l_1^2H_P^3}{3E_PJ_Ps^{(\circ)^3}_{CA} + 2E_{CA}A_{CA}l_1^2H_P^3};$$

$$E_{CA} = \frac{E_S}{1 + \dfrac{\gamma^2_C(l_1^2 + b_T^2)E_S}{12\sigma^{(\circ)^3}_{CA}}};\quad \sigma^{(\circ)}_{CA} = 43,8\left[\frac{\mathrm{KN}}{\mathrm{cm}^2}\right];\quad s^{(\circ)}_{CA} = \sqrt{l_1^2 + b_T^2 + h_P^2};\; E_S = E_T$$

Potential energy of the curtain stays:

$$(V_{int}^{*}+V_{g})_{C} = \sum_{i=1,2}\int_{0}^{L_T}\frac{E_C a_C}{8(s_C^{(\circ)})^3}[(u_T-u_P^2)+(w_T-w_P)^2+(v_{Ci})^2+2h_P(v_{Ci})+2x_{Ci}(w_T-w_P)$$

$$-2(l_1-z)(u_T-u_P)]^2 dz+\sum_{i=1,2}\int_{0}^{L_T}\frac{S_C^{(\circ)}}{2s_C^{(\circ)}}[(u_T-u_P)^2+(w_T-w_P)^2+(v_{Ci})^2]dz$$

with: $S_C^{(\circ)} = \dfrac{g_T s_C^{(\circ)}}{2h_P}$.

Potential energy of the girder:

$$(V_{int}^{*}+V_{g})_{T} = \frac{1}{2}\int_{0}^{L_T}E_T A_T\left\{\frac{\partial u_T}{\partial z}+\frac{1}{2}\left[\left(\frac{\partial u_T}{\partial z}\right)^2+\left(\frac{\partial w_T}{\partial z}\right)^2\right]\right\}^2 dz+\frac{1}{2}\int_{0}^{L_T}E_T J_T\left(\frac{\partial^2 w_T}{\partial z^2}\right)^2 dz$$

$$+\frac{1}{2}\int_{0}^{L_T}E_T J_t\left[\left(\frac{\partial^2 v_{C1}}{\partial z^2}\right)^2+\left(\frac{\partial^2 v_{C2}}{\partial z^2}\right)^2\right]dz+\frac{1}{2}\int_{0}^{L_T}N_T^{(\circ)}\left[\left(\frac{\partial u_T}{\partial z}\right)^2+\left(\frac{\partial w_T}{\partial z}\right)^2\right]dz$$

with: $N_T^{(\circ)} = -\rho_{CA}\dfrac{g_T}{2h_P}[(L_T-l_1)^2-l_1^2]-\dfrac{g_T}{2h_P}[2l_1 z-z^2]$ for

$z\in[0,l_1]$; $N_T^{(\circ)} = -\dfrac{g_T}{2h_P}[(L_T-l_1)^2-(z-l_1)^2]$ for $z\in[l_1,L_T]$.

Potential energy of the pylon:

$$(V_{int}^{*})_P = \frac{1}{2}(K_{Pu}u_P^2+K_{Pw}w_P^2)\ \text{with: } K_{Pu} = \frac{3E_P J_P}{H_P^3};\quad K_{Pw} = \frac{3E_P J_{Pz}}{H_P^3}$$

Kinetic energy of the structure

The kinetic energy of the cable-stayed bridge results from the sum of the contributions given by the stays, the girder and the pylon: $T^{*} = T_{CA}^{*}+T_C^{*}+T_T^{*}+T_P^{*}$.

Kinetic energy of the stays:

$$T_{CA}^{*}+T_C^{*} = +\sum_{i=1,2}\int_{0}^{L_T}\frac{\gamma_C a_C s_C^{(\circ)}}{6g}\left[\left(\frac{\partial w_P}{\partial t}\right)^2+\left(\frac{\partial u_P}{\partial t}\right)^2+\left(\frac{\partial w_T}{\partial t}\right)^2+\left(\frac{\partial u_T}{\partial t}\right)^2+\left(\frac{\partial v_{Ci}}{\partial t}\right)^2+\left(\frac{\partial w_P}{\partial t}\frac{\partial w_T}{\partial t}\right)\right.$$

$$\left.+\left(\frac{\partial u_T}{\partial t}\frac{\partial u_P}{\partial t}\right)\right]dz+\frac{\gamma_C A_{CA} s_{CA}^{(\circ)}}{3g}\left[\left(\frac{\partial w_P}{\partial t}\right)^2+\left(\frac{\partial u_P}{\partial t}\right)^2+\left(\frac{\partial u_{TA}}{\partial t}\right)^2+\left(\frac{\partial u_P}{\partial t}\frac{\partial u_{TA}}{\partial t}\right)\right]$$

Kinetic energy of the girder:

$$T_T^{*} = \frac{1}{2}m_T\int_{0}^{L_T}\left\{\left(\frac{\partial w_T}{\partial t}\right)^2+\left(\frac{\partial u_T}{\partial t}\right)^2+\left(\frac{\partial v_T}{\partial t}\right)^2+\frac{b_T^2}{(z-l_1)^2+h_P^2}\left(\frac{\partial w_T}{\partial t}\right)^2\right\}dz$$

Kinetic energy of the pylon:

$$T_P^* = \frac{1}{2} m_P^{id} \left[\left(\frac{\partial u_P}{\partial t} \right)^2 + \left(\frac{\partial w_P}{\partial t} \right)^2 \right] \text{ with: } m_P^{id} = \frac{33 \gamma_P A_P H_P}{140 g}$$

Virtual work of the viscous damping forces of the structure

The virtual work made by the dissipative forces of the whole structure is given by the sum of the terms related to the stays, the girder and the tower: $\delta W_D = \delta W_{DCA} + \delta W_{DC} + \delta W_{DT} + \delta W_{DP}$.

Virtual work of the viscous damping forces of the stays:

$$\delta W_{DCA} + \delta W_{DC} = -\xi_C c_{cCA} \left\{ \frac{1}{3} s_{CA}^{(\circ)} \left[2 \frac{\partial w_P}{\partial t} \delta(w_P) + 2 \frac{\partial u_P}{\partial t} \delta(u_P) + 2 \frac{\partial u_{TA}}{\partial t} \delta(u_{TA}) + \frac{\partial u_P}{\partial t} \delta(u_{TA}) + \frac{\partial u_{TA}}{\partial t} \delta(u_p) \right] \right\}$$

$$- \sum_{i=1}^{2} \frac{\xi_C}{6} \int_0^{L_T} c_{cC}(z) s_C^{(\circ)}(z) \begin{bmatrix} 2 \frac{\partial w_P}{\partial t} \delta(w_P) + 2 \frac{\partial u_P}{\partial t} \delta(u_p) + 2 \frac{\partial w_T}{\partial t} \delta(w_T) + 2 \frac{\partial v_{Ci}}{\partial t} \delta(v_{Ci}) + \\ + 2 \frac{\partial u_T}{\partial t} \delta(u_T) + \frac{\partial w_P}{\partial t} \delta(w_T) + \frac{\partial w_T}{\partial t} \delta(w_P) + \frac{\partial u_P}{\partial t} \delta(u_T) + \frac{\partial u_T}{\partial t} \delta(u_P) \end{bmatrix} dz$$

where: ξ_C is the damping ratio of the stays; $c_{cCA} = 2 \frac{\gamma_C A_{CA}}{g} \omega_{nCA}$ and $c_{cC} = 2 \frac{\gamma_C a_C}{g} \omega_{nC}$ are the critical dampings per length unit concerning respectively the anchorage stay and the generic curtain stay, being ω_{nCA} and ω_{nC} their corresponding natural frequencies.

Virtual work of the viscous damping forces of the girder:

$$\delta W_{DT} = -\xi_T \int_0^{L_T} c_{cT} \left\{ \frac{\partial u_T}{\partial t} \delta(u_T) + \frac{\partial v_T}{\partial t} \delta(v_T) + \frac{\partial w_T}{\partial t} \delta(w_T) \left[1 + \frac{b_T^2}{h_P^2 + (z - l_1)^2} \right] \right\} dz$$

where: ξ_T is the damping ratio of the girder; $c_{cT} = 2\, m_T \omega_{nT}$ is the critical damping per length unit of the girder and ω_{nT} is its natural frequency.

Virtual work of the viscous damping forces of the tower:

$$\delta W_{DP} = -\xi_p \left[c_{cPu}^{id} \frac{\partial u_P}{\partial t} \delta(u_P) + c_{cPw}^{id} \frac{\partial w_P}{\partial t} \delta(w_P) \right]$$

where: ξ_P is the damping ratio of the tower; $c_{cPu}^{id} = 2 m_P^{id} \omega_{nPu}$ and $c_{cPw}^{id} = 2 m_P^{id} \omega_{nPw}$ are the ideal critical dampings per length unit of the tower, respectively in the directions z and x of the global reference system, while ω_{nPu} and ω_{nPw} are its natural frequencies in the same directions.

Appendix C: System of differential equations

$$-\frac{A_{CA}\gamma_C}{3g}s_{CA}^{(\circ)}\left[2\frac{\partial^2 u_P}{\partial t^2}+\frac{\partial^2 u_{TA}}{\partial t^2}\right]-\frac{\gamma_C}{3g}\int_0^{L_T}a_C(z)s_C^{(\circ)}(z)\left[2\frac{\partial^2 u_P}{\partial t^2}-C_U(z)\left[\left(\frac{\partial w_T}{\partial t}\right)^2+w_T\frac{\partial^2 w_T}{\partial t^2}\right]\right]dz$$

$$-m_P^{id}\left(\frac{\partial^2 u_P}{\partial t^2}\right)-\frac{E_{CA}A_{CA}}{(s_{CA}^{(\circ)})^3}[3(u_{TA}-u_P)^2 l_1+w_P^2 l_1-2l_1^2(u_{TA}-u_P)]+\frac{2S_{CA}^{(\circ)}}{s_{CA}^{(\circ)}}(u_{TA}-u_P)$$

$$+\int_0^{L_T}\frac{E_C(z)a_C(z)}{(s_C^{(\circ)}(z))^3}\Big[3u_P^2(z-l_1)+\left(\frac{C_V(z)}{h_P}\right)^2 b_T^2 w_T^2(z-l_1)+(w_T-w_P)^2(z-l_1)-2(z-l_1)^2\left(C_U(z)\frac{w_T^2}{2}+u_P\right)$$

$$-C_V(z)w_T^2(z-l_1)\Big]dz-\int_0^{L_T}\frac{2S_C^{(\circ)}(z)}{s_C^{(\circ)}(z)}\left[C_U(z)\frac{w_T^2}{2}+u_P\right]dz-K_{Pu}u_P-\frac{\xi_c c_{cCA}}{3}s_{CA}^{(\circ)}\left[2\frac{\partial u_P}{\partial t}+\frac{\partial u_{TA}}{\partial t}\right]$$

$$-\frac{\xi_C}{3}\int_0^{L_T}c_{cC}(z)s_C^{(\circ)}(z)\left[2\frac{\partial u_P}{\partial t}-C_U(z)w_T\frac{\partial w_T}{\partial t}\right]dz-\xi_P c_{cPu}^{id}\left(\frac{\partial u_P}{\partial t}\right)=0 \qquad (1)'$$

$$-\frac{2A_{CA}\gamma_C}{3g}s_{CA}^{(\circ)}\frac{\partial^2 w_P}{\partial t^2}-\frac{\gamma_C}{3g}\int_0^{L_T}a_C(z)s_C^{(\circ)}(z)\left[2\frac{\partial^2 w_P}{\partial t^2}+\frac{\partial^2 w_T}{\partial t^2}\right]dz-m_P^{id}\frac{\partial^2 w_P}{\partial t^2}-\frac{E_{CA}A_{CA}}{(s_{CA}^{(\circ)})^3}\;[-2l_1(u_{TA}-u_P)w_P$$

$$+2b_T^2 w_P]-\frac{2S_{CA}^{(\circ)}}{s_{CA}^{(\circ)}}w_P+\int_0^{L_T}\frac{E_C(z)a_C(z)}{(s_C^{(\circ)}(z))^3}\begin{bmatrix}-2(z-l_1)(w_T-w_P)u_P-2C_V(z)b_T^2 w_T+\\+2b_T^2(w_T-w_P)\end{bmatrix}dz$$

$$+\int_0^{L_T}\frac{2S_C^{(\circ)}(z)}{s_C^{(\circ)}(z)}(w_T-w_P)dz-K_{Pw}w_P-\frac{2\xi_C c_{cCA}}{3}s_{CA}^{(\circ)}\frac{\partial w_P}{\partial t}-\frac{\xi_C}{3}\int_0^{L_T}c_{cC}(z)s_C^{(\circ)}(z)\left[2\frac{\partial w_P}{\partial t}+\frac{\partial w_T}{\partial t}\right]dz$$

$$-\xi_P c_{cPw}^{id}\frac{\partial w_P}{\partial t}=0 \qquad (2)'$$

$$-\frac{A_{CA}\gamma_C}{3g}s_{CA}^{(\circ)}\left[2\frac{\partial^2 u_{TA}}{\partial t^2}+\frac{\partial^2 u_P}{\partial t^2}\right]-\frac{E_{CA}A_{CA}}{(s_{CA}^{(\circ)})^3}\begin{bmatrix}-3(u_{TA}-u_P)^2 l_1-w_P^2 l_1+\\+2l_1^2(u_{TA}-u_P)\end{bmatrix}$$

$$-\frac{2S_{CA}^{(\circ)}}{s_{CA}^{(\circ)}}(u_{TA}-u_P)-\frac{\xi_C c_{cCA}}{3}s_{CA}^{(\circ)}\left[2\frac{\partial u_{TA}}{\partial t}+\frac{\partial u_P}{\partial t}\right]=0; \qquad (3)'$$

$$-\frac{\gamma_C}{3g}a_C s_C^{(\circ)}(z)\left\{\begin{array}{l}\left[2\frac{\partial^2 w_T}{\partial t^2}+\frac{\partial^2 w_P}{\partial t^2}\right]+2\left(\frac{C_V(z)}{h_P}\right)^2\left[w_T\left(\frac{\partial w_T}{\partial t}\right)^2+(b_T^2+w_T^2)\frac{\partial^2 w_T}{\partial t^2}\right]+\\ -\frac{\partial^2 u_P}{\partial t^2}C_U(z)w_T+2(C_U(z))^2\left[w_T\left(\frac{\partial w_T}{\partial t}\right)^2+w_T^2\frac{\partial^2 w_T}{\partial t^2}\right]\end{array}\right\}-E_T J_T\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2 w_T}{\partial z^2}\right)$$

$$-m_T\frac{\partial^2 w_T}{\partial t^2}-m_T\left[\left[(C_U(z))^2+\left(\frac{C_V(z)}{h_P}\right)^2\right]\left[w_T\left(\frac{\partial w_T}{\partial t}\right)^2+w_T^2\frac{\partial^2 w_T}{\partial t^2}\right]+b_T^2\frac{C_V(z)}{h_P^2}\frac{\partial^2 w_T}{\partial t^2}\right]-\frac{2S_C^{(\circ)}(z)}{s_C^{(\circ)}(z)}\Big[1+C_U(z)u_P$$

$$+\left(\frac{C_V(z)}{h_P}\right)^2 b_T^2\Big]w_T+\frac{2S_C^{(\circ)}(z)}{s_C^{(\circ)}(z)}w_P+\frac{\partial}{\partial z}\left[N_T^{(\circ)}(z)\left(\frac{\partial C_U(z)}{\partial z}\frac{w_T^2}{2}+C_U(z)w_T\frac{\partial w_T}{\partial z}\right)\right]C_U(z)w_T$$

$$+\frac{\partial}{\partial z}\left[N_T^{(o)}(z)\frac{\partial w_T}{\partial z}\right]-\frac{E_C a_C}{(s_C^{(\circ)}(z))^3}[-2C_V(z)b_T^2 w_T+2(z-l_1)^2 C_U(z)w_T u_P-2(z-l_1)\left(\frac{C_V(z)}{h_P}\right)^2 b_T^2 w_T u_P$$

$$+2(z-l_1)C_V(z)w_T u_P+2(C_V(z))^2 b_T^2 w_T-2C_V(z)b_T^2(w_T-w_P)-2(z-l_1)(w_T-w_P)u_P$$

$$+2b_T^2(w_T-w_P)]-\xi_T c_{cT}\left[1+(C_U(z))^2 w_T^2+\left(\frac{C_V(z)}{h_P}\right)^2 w_T^2+b_T^2\frac{C_V(z)}{h_P^2}\right]\frac{\partial w_T}{\partial t}$$

$$-\frac{\xi_C}{3}c_{cC}s_C^{(\circ)}(z)\left[\frac{\partial w_P}{\partial t}-\frac{\partial u_P}{\partial t}CU(z)w_T\right]-\frac{\xi_C}{3}c_{cC}s_C^{(\circ)}(z)\left[2+2\left(\frac{C_V(z)}{h_P}\right)^2(b_T^2+w_T^2)+2(C_U(z))^2 w_T^2\right]\frac{\partial w_T}{\partial t}$$

$$+E_T A_T\left\{\begin{array}{l}-\frac{\partial}{\partial z}\left[\frac{\partial w_T}{\partial z}\left(\frac{\partial C_U(z)}{\partial z}\frac{w_T^2}{2}+C_U(z)w_T\frac{\partial w_T}{\partial z}\right)\right]+\frac{\partial}{\partial z}\left[\frac{\partial C_U(z)}{\partial z}\frac{w_T^2}{2}+C_U(z)w_T\frac{\partial w_T}{\partial z}\right]C_U(z)w_T+\\ -\frac{3}{2}\frac{\partial}{\partial z}\left[\frac{\partial C_U(z)}{\partial z}\frac{w_T^2}{2}+C_U(z)w_T\frac{\partial w_T}{\partial z}\right]^2 C_U(z)w_T-\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial w_T}{\partial z}\right)^2 C_U(z)w_T\end{array}\right\}+$$

$$-E_T J_t\left\{\frac{\partial^2}{\partial z^2}\left\{\frac{C_V(z)}{h_P}\left[\left(\frac{\partial w_T}{\partial z}\right)^2+(b_T+w_T)\frac{\partial^2 w_T}{\partial z^2}\right]+\frac{2}{h_P}\frac{\partial C_V(z)}{\partial z}(b_T+w_T)\frac{\partial w_T}{\partial z}\right\}\times\frac{C_V(z)}{h_P}(b_T+w_T)\right\}+$$

$$-E_T J_t\left\{\frac{\partial^2}{\partial z^2}\left\{\frac{C_V(z)}{h_P}\left[\left(\frac{\partial w_T}{\partial z}\right)^2+(w_T-b_T)\frac{\partial^2 w_T}{\partial z^2}\right]+\frac{2}{h_P}\frac{\partial C_V(z)}{\partial z}(w_T-b_T)\frac{\partial w_T}{\partial z}\right\}\times\frac{C_V(z)}{h_P}(w_T-b_T)\right\}=0$$

(4)'

Appendix D: Program *"main1"*

```
Clear
format long e
l1 = 147.42;
LT = l1 + 210.6;
K = (LT – l1)/200;
z = 0 : K : l1 ;
N = length (z) – 1;
z1 = l1 + K : K : LT;
M = length [z1] ;
z = [z z1] ;
A = zeros (2*(N + M + 1) + 1);
K2 = K^2;
ω = input("Write the value of ω");
for i = 1 : N – 2
j = 2*i – 1;
A([ j j + 1], [j j + 1]) = 2*(K2*alfa 3(z(i), ω) – 2*alfa 1(z(i)));
A([ j j +1], [j + 2 j + 3]) = 2*alfa 1(z(i)) + K*alfa 2(z(i));
A([ j + 2 j + 3], [j j + 1] = 2*alfa 1(z(i+1)) – K*alfa 2(z(i + 1));
end
A([2*N – 3 2*N – 2], [2*N – 3 2*N – 2]) = 2*(K2*alfa 3(z(N –1), ω) – 2*alfa 1(z(N – 1)));
A(2*N – 3,2*N – 1) = 2*A1(z(N – 1)) + K*B1(z(N – 1));
A(2*N – 1,2*N – 3) = – B1(z(N));
A(2*N – 1,2*N – 2) = – D1(z(N));
A(2*N – 1,2*N – 1) = B1(z(N)) + K*C1(z(N));
A(2*N – 1,2*N) = K*A1(z(N));
A(2*N, 2*N – 1) = K*C2(z(N)) – B2(z(N));
A(2*N, 2*N + 1) = K*A2(z(N));
A(2*N, 2*N + 2) = B2(z(N));
A(2*N, 2*N + 3) = D2(z(N));
A(2*N + 1, 2*N – 2) = 1;
A(2*N + 1, 2*N + 3) = 1;
A(2*N + 2, 2*N– 1) = 2*A2(z(N + 1)) – K*B2(z(N + 1));
for i = N +1 : M + N
j = 2*i – 1;
A([ j + 1 j + 2], [j + 1 j + 2]) = 2*(K2*beta 3(z(i), ω) – 2*beta 1(z(i)));
A([ j + 1 j + 2], [ j + 3 j + 4]) = 2*beta 1(z(i)) + K*beta 2(z(i));
A([ j + 3 j + 4], [ j + 1 j + 2]) = 2*beta 1(z(i + 1)) – K*beta 2(z(i +1));
end
j = 2*(M + N + 1) – 1;
A([j + 1 j + 2], [j + 1 j + 2] = 2*(K2 beta 3(z(M + N + 1), ω) – 2*beta 1(z(M + N + 1)));
[U, D] = eig(A);
[λmin, KK] = min(abs(diag(D)));
x = U(:, KK);
λmin
x2 = x(length(x) : – 2 : length(x) – 2*M + 3);
x1 = [0; x(2 : 2 : 2*(N – 1)); 0; x2(length(x2) : –1 : 1); 0];
g = [0 : LT/750 : LT];
y = spline(z(1 : length(x1)), x1, g);
plot(g, y)
```

CC