Computers and Concrete, *Vol.13, No.4 (2014)457-482*
DOI: http://dx.doi.org/10.12989/cac.2014.13.4.457

Optimal design of reinforced concrete beams: A review

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(*Received May 31, 2013, Revised January 21, 2014, Accepted February 1 2014*)

Abstract. This paper summarizes available literature on the optimization of reinforced concrete (RC) beams. The objective of optimization (e.g. minimum cost or weight), the design variables and the constraints considered by different studies vary widely and therefore, different optimization methods have been employed to provide the optimal design of RC beams, whether as isolated structural components or as part of a structural frame. The review of literature suggests that nonlinear deterministic approaches can be efficiently employed to provide optimal design of RC beams, given the small number of variables. This paper also presents spreadsheet implementation of cost optimization of RC beams in the familiar MS Excel environment to illustrate the efficiency of the exhaustive enumeration method for such small discrete search spaces and to promote its use by engineers and researchers. Furthermore, a sensitivity analysis is performed on the contribution of various design parameters to the variability of the overall cost of RC beams.

Keywords: reinforced concrete beam; optimization; discrete search space; exhaustive enumeration; genetic algorithm; spreadsheet implementation; sensitivity analysis

1. Introduction

Reinforced concrete (RC) is now widely used in a variety of structures owing to its versatility, high compressive strength, durability and resistance to fire and water damage. The vast usage of concrete structures calls for economical design, and thus, many attempts have been made to optimize the structural design of RC structures (Structural Engineering Institute 2002), including the fundamental design of RC beams as primary bending elements.

Conventional structural design of steel reinforced concrete beams involves iterative design and checks for section dimensions and the amount of steel reinforcement. The process usually starts with a trial section, where the depth of the beam is selected based on guidelines for deflection control. The composite flexural resistance of the trial section is checked against the applied bending moment, considering the effects of the self-weight. This is then followed by checks for shear resistance, deflection and other code requirements. This practice usually requires many repeats and takes considerable time until a section is found that satisfies both ultimate (strength) and serviceability limit states prescribed by design codes. The resultant design, while complying

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http://www.techno-press.org/?journal=cac&subpage=8ISSN: 1598-8198(Print), 1598-818X (Online)

with all the code requirements, by no means provides an optimal solution.

Advancement in computer technology and analysis has led many researchers to develop modern techniques for economic design of RC members and structures, yet early research studies on the optimization of RC beams date back to 1960s (Norman 1964), when access to machine computing was very limited. For that reason and many others, the objective of the optimization, the restrictions applied and the methods employed to find the optimized solution has varied widely among different research works. This paper, therefore, aims to provide a review of the literature on the structural design optimization of steel reinforced concrete beams. The problem formulation by different researchers and various optimization techniques applied to RC beam design are studied and presented in the following sections. Finally, a simple Excel based optimization tool is developed and illustrated with an example.

2. Problem formulation

The flexural design procedure for reinforced concrete beams now established in most building codes around the world, including the Eurocode 2 (British Standards Institution, 2007), ACI 318-05 (American Concrete Institute 2005) and CSA A23.3-04 (Canadian Standards Association 2004), is based on the ultimate strength design method. This procedure accounts for the nonlinear material properties, and recognizes the difference in certainty of various load types (e.g. dead and live loads) by considering different safety factors for the load types and their combinations. Based on this design method, RC beam sections have to satisfy ultimate (strength) and serviceability limit state criteria, which, respectively, account for the ultimate failure of the structure and its functionality for the intended routine use.

While the design criteria for steel reinforced concrete beams are well established, the optimal design problem requires to be clearly defined by its objective(s), design variables and the constraints enforced by codes or prompted by practical restrictions. A typical optimization problem can be formulated mathematically as (Structural Engineering Institute 2002):

$$\text{Minimize (or maximize) } f(x)$$

$$Subject\ to \quad \begin{cases} g_j(x) = 0\ ,\ \ j = 1, \dots, p \\ g_j(x) \leq 0\ ,\ \ j = p+1, \dots, m \end{cases} \tag{1}$$

$$x = [x_1, \dots, x_n]$$

$$x_{iL} \leq x_i \leq x_{iU}\ , \qquad i = 1, \dots, n$$

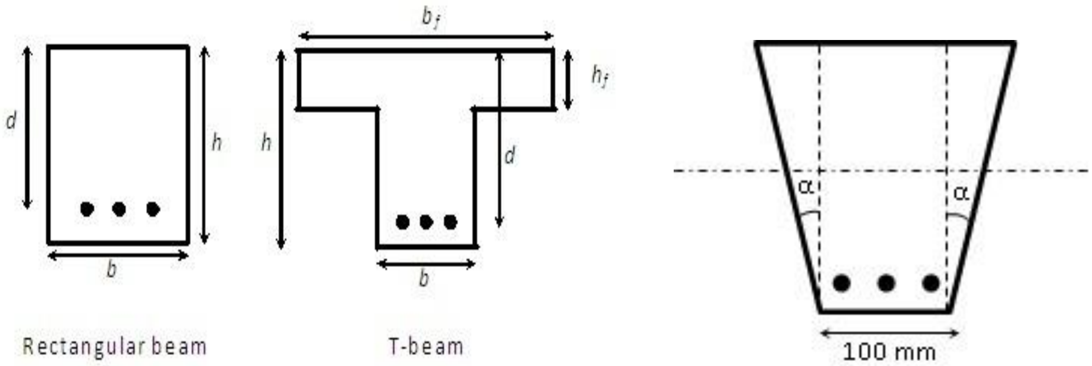
where $f(x)$ is the objective function, which may require minimization (cost, weight, etc.) or maximization (benefits); $g_i(x)$ is an equality or inequality constraint; x_{iL} and x_{iU} are the lower and upper bounds for the design variable x_i; and n, m and p are the number of design variables, total constraints and equality constraints, respectively.

For the optimization problem of a RC beam design, various research studies over the years have adopted different design variables as well as objective functions and constraints. The following sections present a summary of such problem formulations.

2.1 Design variables

The design variables in optimization of RC beams are generally associated with the dimensions of the beam and the area of steel reinforcement bars (tensile, compressive, and shear reinforcement) as well as their arrangement (see Table 1). Although rarely adopted, the strength of the materials can also be considered as variables (Goble and Moses 1975). The number of variables considered in various research studies are normally limited to a handful. In a very early study, Norman (1964) chose the depth of the beam as the only design variable for singly reinforced rectangular and T-beams, while more than 30 years later, Rajeev and Krishnamoorthy (1998) envisaged seven design variables that included the width and depth of the beam and the area of five reinforcement groups to allow for a non-uniform distribution of tensile and compressive reinforcement along the length of the beam. Nevertheless, the main variables are typically one or both dimensions of a rectangular beam and the area of tensile reinforcement (Chakrabarty 1992b, Camp *et al*. 2003).The depth of the beam is either expressed as the effective depth (d) (i.e., depth of tensile reinforcement from the top compression edge) or as the overall height of the beam (h), as shown in Fig. 1(a). Nonetheless, the thickness of concrete cover can be assumed fixed and dictated by durability and other construction requirements. For T-beams, it is usually assumed that the thickness of the flange (hf) is determined by the depth of the floor slab and therefore assumed to be fixed (Chou 1977, Balaguru 1980b, Prakash *et al*. 1988).

Rectangular or T-beams are the most commonly practiced shapes; however, in a recent study Narayan and Venkataramana (2007) attempted to optimize the shape of a RC beam. Referring to the pioneer study in shape optimization by Michell (1904), they argued that shape should be considered as a primary variable in design of structures. Due to concrete cracking, RC design approaches ignore the insignificant strength of concrete in tension zone. Thus, to reduce the concrete in tensile zone and incorporate it in the more efficient compression zone, Narayan and Venkataramana assumed a trapezoidal shape for the RC beam (Fig. 1(b)). They adopted the depth of the beam and α (the angle in Fig. 1(b)) as the design variables. Comparing the costs of rectangular and trapizoidal beams, Narayan and Venkataramana concluded that trapezoidal RC beams are more economical and their usage should be encouraged.



(a) Rectangular and T-beams

(b) Trapezoidal beam assumed by Narayan and Venkataramana (2007)

Fig. 1 Cross-sectional dimensions of reinforced concrete beams

Table 1 Design variables adopted by various research studies

Study	**Design variables**
Norman (1964)	D
Sandhu (1971)	A_s
Friel (1974)	D, A_s
Goble and Moses (1975)	b, D, A_s - also suggest provisions for f_c
Chou (1977)	T beams – D and A_s
Balaguru (1980a)	b, D, A_s, A'_s
Balaguru (1980b)	T beams – b_w, D and A_s
Colin and Macrae (1984)	A_s and A'_s for RC beams of I, T or rectangular shapes
Prakash *et al*. (1988)	D/b ratio and A_s. D is assumed to be fixed.
Chakrabarty (1992a)	b, D, A_s
Chakrabarty (1992b)	b, D, A_s
Adamu and Karihaloo (1994b)	D, A_s (A_s is allowed to change freely along the beam)
Adamu and Karihaloo (1994a)	D, A_s (each variable is allowed to change freely along the beam)
Adamu and Karihaloo (1994c)	b, D, A_s (each variable is allowed to change freely along the beam)
Chung and Sun (1994)	b and A_s
Al-salloum and Siddiqi (1995)	D and A_s
Coello and Hernández (1997)	b, D, A_s
Rajeev and Krishnamoorthy (1998)	b, D and 4 other variables for area of steel reinforcement along a continuous beam: Top and bottom continuous bar diameter, additional top reinforcement at support, and additional bottom reinforcement at mid-span
Ceranic and Fryer (2000)	D, A_s
Dole *et al*. (2000)	D, A_s
Camp *et al*. (2003)	b, D, A_s, A'_s
Ferreira *et al*. (2003)	Ratio of A'_s/A_s (Although not declared as variable, considered 8 cases for the position of neutral axis and depth of the flange of T-beam.)
Lee and Ahn (2003)	b, D, A_s, A'_s
Lepš and Šejnoha (2003)	b, D, A_s, A'_s
Guerra and Kiousis (2006)	b, D, A_s, A'_s
Narayan and Venkataramana (2007)	D and angle α in Fig. 1(b).
González-Vidosa *et al*. (2008)	b, D and other variables for area of steel reinforcement along a continuous beam: Top and bottom continuous bar diameter, additional top reinforcement at left and right supports, and additional bottom reinforcement at mid-span

D: beam depth (h or d)
b: beam width
b_w: T-beam web width
A_s: area of tensile steel reinforcement
A'_s: area of compressive steel reinforcement
f_c: Concrete strength

Table 2 Optimization objective function as adopted by various research studies

Study	Optimization objective	Included in the objective function
Norman (1964)	Minimum cost	C_s, C_c and C_f
Sandhu (1971)	Minimum cost	C_s, C_c and C_f
Friel (1974)	Minimum cost	C_s, C_c, C_f and the cost of increase in the building height due to beam depth
Goble and Moses (1975)	Minimum cost	C_s, C_c and C_f
Chou (1977)	Minimum cost	C_s and C_c
Balaguru (1980a)	Minimum cost	C_s, C_c and C_f
Balaguru (1980b)	Minimum cost	C_s, C_c and C_f
Colin and Macrae (1984)	Minimum cost Also suggest alternative objectives such as minimum weight, concrete volume or steel reinforcement	C_s, C_c and C_f
Prakash *et al*. (1988)	Minimum cost	C_s and C_c
Chakrabarty (1992a)	Minimum cost	C_s, C_c and C_f
Chakrabarty (1992b)	Minimum cost	C_s, C_c and C_f
Adamu and Karihaloo (1994b)	Minimum cost	C_s, C_c and C_f
Adamu and Karihaloo (1994a)	Minimum cost	C_s, C_c and C_f
Adamu and Karihaloo (1994c)	Minimum cost	C_s, C_c and C_f
Chung and Sun (1994)	Minimum weight	Weight of concrete and steel
Al-salloum and Siddiqi (1995)	Minimum cost	C_s, C_c and C_f
Coello and Hernández (1997)	Minimum cost	C_s, C_c and C_f
Rajeev and Krishnamoorthy (1998)	Minimum cost	C_s, C_c and C_f
Ceranic and Fryer (2000)	Minimum cost	C_s and C_c
Dole *et al*. (2000)	Minimum cost	C_s and C_c
Camp *et al*. (2003)	Minimum cost	C_s, Cc and C_f
Ferreira *et al*. (2003)	Minimum Reinforcement	Area of steel reinforcement
Lee and Ahn (2003)	Minimum cost	C_s, C_c and C_f
Lepš and Šejnoha (2003)	Minimum cost	C_s and C_c
Guerra and Kiousis (2006)	Minimum cost	C_s, C_c, cost of formworks as a function of beam width and cost of placing concrete and vibrating, including labour and equipment as a function of cross-sectional area
Narayan and Venkataramana (2007)	Minimum cost	C_s, C_c and C_f
González-Vidosa *et al*. (2008)	Minimum cost	C_s, C_c and C_f

C_s: Cost of steel - constant per weight
C_c: Cost of concrete - constant per weight
C_f: Cost of formworks - constant per surface area

2.2 Optimization objectives

Design optimization of structures is evidently driven by economic implications of building construction. Yet, the objective function for design optimization of RC beams is defined differently by various researchers (see Table 2). Chung and Sun (1994) for example, adopted minimization of the overall weight of the beam as the objective. Acknowledging that weight does not appropriately represent the cost of material, they performed a comparative study with three ratios for cost per unit weight of concrete (C_c) to that of steel (C_s). The maximum cost reduction was obtained for the case where the cost of concrete was ten times compared to steel. However, when steel was assumed to cost ten times compared to concrete, the optimum cost showed an increase and the upper limit of the beam width was reached, implying that, owing to the high C_s, increasing the width of concrete beam did not sufficiently increase the resistance of the section. Nevertheless, realistic cost ratios of concrete and steel are far from Chung and Sun's assumptions, with C_s being of the order of a few tens of C_c (Camp *et al*. 2003).

Naturally, the majority of the available literature has adopted cost minimization as the objective of optimal design. Some research works only included the cost of concrete and steel in the analysis (Chou 1977, Prakash *et al*. 1988, Ceranic and Fryer 2000, Lepš and Šejnoha 2003). Prakash *et al*. (1988) argued that a weight-base optimization is better suited to high-rise buildings where the same component recurs in all stories; nonetheless they used the minimum cost criterion as the basis of their optimization approach. They stated that the cost of formwork (C_f) is also one of the main factors to be considered, but they neglected it in calculations to simplify the problem. To cover the practical variety, Prakash *et al*. considered the cost ratio of the unit volume of steel to concrete to vary in the range 50 to 100.

Most other studies accounted for the cost of formworks as well as the material, assuming constant values for the cost per unit weight of material and per unit area of formwork (Norman 1964, Sandhu 1971, Goble and Moses 1975, Balaguru 1980b, Al-salloum and Siddiqi 1995). Friel (1974) also considered in the objective function, the cost of the increase in the building height due to the beam depth. In a more recent study, Guerra and Kiousis (2006) incorporated the cost of placement, labour, equipment and accessories in the construction costs in addition to the material costs. Moreover, they defined the cost of forming and placing concrete as a function of cross-sectional dimensions, introducing a non-linear cost coefficient in the objective function.

2.3 Constraints

Optimization constraints are the functional and structural requirements of the structure expressed as equality or inequality equations. Design constraints for RC beams can either be induced by practical restrictions, for example on constructability and transportability, that limit the design variables, or be enforced by code requirements on structural response of the structure (see Table 3). As mentioned before, the code requirements on a RC beam are of two types: ultimate limit states (ULS) -also called strength limit states- and serviceability limit states (SLS). As outlined below, there are three main flexural ULS requirements for a RC beam.

1. The moment resistance capacity of the cross-section should be higher than the applied bending moment. The applied bending moment generally includes the effects of the self-weight of the beam. The bending moment capacity imposes a nonlinear constraint to the optimization and can be expressed as

Table 3 Optimization constraints as adopted by various research studies

Study	Constraints
Norman (1964)	A_s is at balanced design, b is constant
Sandhu (1971)	X1
Friel (1974)	X1, X2, X3
Goble and Moses (1975)	Not reported
Chou (1977)	X1, X2, X3
Balaguru (1980a)	X1, X2, X3
Balaguru (1980b)	X1, X4, reinforcement ratio is fixed at maximum allowable
Colin and Macrae (1984)	X1, X2, X3, X11
Prakash *et al*. (1988)	X1, X3, X4
Chakrabarty (1992a)	X1, X5
Chakrabarty (1992b)	X1, X6
Adamu and Karihaloo (1994b)	X1, X2, X3, X8, X9, X11, X12
Adamu and Karihaloo (1994a)	X1, X2, X3, X8, X9, X11, X12
Adamu and Karihaloo (1994c)	X1, X2, X3, X6, X7, X8, X9, X11, X12
Chung and Sun (1994)	X1 (expressed in terms of tensile and compressive stresses), X2, X3, X6, X7
Al-salloum and Siddiqi (1995)	X1, X2, X3, X9 and suggested but not used X8
Coello and Hernández (1997)	X1, X4, X5, X6
Rajeev and Krishnamoorthy (1998)	X1, X2, X3, X4, X5, X6, X7, X8, X9, X10
Ceranic and Fryer (2000)	X1, X2, X3
Dole *et al*. (2000)	X1, X2 (expressed as maximum depth of the neutral axis) and maximum concrete cover which is checked after optimization is performed.
Camp *et al*. (2003)	X1, X2, X3, X6 (to allow minimum reinforcement clear spacing), X5, X8 (to control deflection), X9, X12 and maximum difference in bar sizes within a single row of reinforcement
Ferre ira *et al*. (2003)	X1, X3
Lee and Ahn (2003)	X1, X2, X3, X4, X5, X6, X7 and allowable arrangements of bars in two rows
Lepš and Šejnoha (2003)	X1, X11, X12
Guerra and Kiousis (2006)	X1, X2, X3, X4, X5, X6 (to allow minimum reinforcement clear spacing), X7, X8, X9 and tensile reinforcement area is greater that the compressive reinforcement area
Narayan and Venkataramana (2007)	X1, X2, X3, X9, X11 (assuming the effective moment inertia of the section is half of the gross moment of inertia) and minimum and maximum values for the angle of the trapezoidal beam
González-Vidosa *et al*. (2008)	X1, X11, X12

X1: Moment resistance capacity constraint
X2: Minimum steel reinforcement
X3: Maximum steel reinforcement
X4: Minimum beam width to depth ratio
X5: Minimum beam width to depth ratio
X6: Minimum beam width
X7: Maximum beam width
X8: Minimum beam depth
X9: Maximum beam depth
X10: Maximum beam span to depth ratio
X11: Maximum allowable deflection
X12: Shear Capacity constraint

$$M_f + M_{sw} \leq M_r \tag{2}$$

where M_f is the factored design bending moment, M_{sw} is the bending moment due to self-weight of the beam and M_r is the factored bending moment resistance

$$M_r = \varphi_s f_y A_s \left(d - \frac{a}{2}\right) \tag{3}$$

$$a = \frac{\varphi_s f_y A_s}{\alpha_1 \varphi_c f_c^{'} b} \tag{4}$$

where

- φ_s is the resistance factor for steel,
- φ_c is the resistance factor for concrete,
- f_y is the yield strength of steel reinforcement,
- $f_c^{'}$ is the compressive strength of concrete,
- A_s is the area of tensile steel reinforcement,
- b is the width of the beam,
- d is the effective depth of the beam,
- a is the depth of the equivalent rectangular stress block, and
- α_1, β_1 are the equivalent rectangular stress distribution parameters.

2. The area of tension reinforcement (A_s) should satisfy a minimum amount to ensure that the steel reinforcement compensates the loss of tensile strength caused by cracking in the concrete. This nonlinear constraint can be expressed as

$$A_s \geq \frac{0.2\sqrt{f_c^{'}}}{f_y} bh \tag{5}$$

where h is the overall height of the beam and $A_{s,min} = \frac{0.2\sqrt{f_c^{'}}}{f_y} bh$ is the minimum reinforcement provisioned by CSA A23.3-04 (Canadian Standards Association 2004).

3. The area of tension reinforcement (A_s) should satisfy a maximum amount to avoid brittle concrete-controlled failure. This constraint is also nonlinear and provided in Eq. (6).

$$\left(\frac{0.75 \times 700}{700 + f_y}\right)\left(\frac{\alpha_1 \beta_1 \varphi_c f_c^{'}}{\varphi_s f_y}\right) bd \tag{6}$$

where $A_{s,max} = \left(\frac{0.75 \times 700}{700 + f_y}\right)\left(\frac{\alpha_1 \beta_1 \varphi_c f_c^{'}}{\varphi_s f_y}\right) bd$ is the maximum reinforcement provisioned by CSA A23.3-04.

Serviceability limit state requirements are generally of two types: restrictions on maximum deflection of the beam and provisions to control the crack width in concrete. Due to formation of cracks in concrete, calculating the deflection of RC beams can be tedious. To avoid the cumbersome calculations, design codes (e.g. CSA A23.3-04) allow for an alternative indirect approach for deflection control. This approach sets an upper limit on the span to depth ratio of the beam and ensures sufficient stiffness in the beam to control the deflection. The indirect deflection control is commonly adopted for applications in usual ranges of load and member sizes. The direct approach however imposes a complex nonlinear constraint.

Practical restrictions generally limit the dimensions of a RC beam, imposing minimum and maximum bounds for both width and depth of the beam. Moreover, detailing requirements usually set a minimum spacing between the reinforcement bars and therefore dictate more feasibility constraints on the design variables.

Flexural and shear optimization of a RC beam can be done separately one after another. With the beams in regular range, flexure dominates the design and should be optimized first. For deep beams the procedure might be reversed. Nevertheless, even when flexural design precedes shear design, there is an ultimate limit state constraint corresponding to the shear design of the beam that can affect the flexural optimization and should be taken into account. This constraint sets a maximum limit for the overall factored shear resistance of the section and is prescribed to prevent brittle shear compression failure especially in beams heavily reinforced with shear reinforcement. This ultimate limit state essentially adds a lower bound limit to the width of the section.

2.4 Continuous and discrete search space

In analytical design of RC beams, for calculation purposes, the design variables (i.e., cross-sectional dimensions and reinforcements) are assumed to be continuous quantities. However, the final dimensions and reinforcements are selected from among practical discrete choices. The cross-sectional dimensions of concrete beams are rounded to nominal sizes as a result of practical limits on the accuracy of formwork measurements and construction. Therefore, although optimization of concrete beams can be performed using continuous beam dimensions, the final selection of dimensions will be from a discrete set of nominal sizes based on a pragmatic accuracy. The same applies to selection of steel reinforcements. Flexural constraints on the RC beam limit the minimum and maximum area of reinforcement. However, to provide the area required, the reinforcement bars are selected from discrete nominal sizes available in the construction industry. Moreover, the detailing requirements of the design codes restrict the clear spacing between the reinforcement. To satisfy such requirements, the mere area of the reinforcement would not be sufficient; the size and number of the reinforcement bars should be known.

Thus, the optimization search space for RC beams is essentially discrete. However, given the lower complexity of some continuous optimization techniques the discrete nature of the design variables is ignored by many early research studies and it is assumed that treating the design variables as continuous leads to sufficiently accurate optimal results (Norman 1964, Sandhu 1971, Friel 1974, Chou 1977, Balaguru 1980a, Prakash *et al*. 1988, Chakrabarty 1992a, Chung and Sun 1994, Al-salloum and Siddiqi 1995).

3. Optimization techniques

Once the optimization problem is fully defined by its objective(s) and constraints, a suitable method can be chosen to find the optimal solution. A vast range of optimization techniques are available that can be categorized into two main types: linear and non-linear programming techniques.

Linear programming approaches can be applied to problems where the objective functions and constraints can all be expressed by linear equations. The most widely used algorithm for linear programming problems with a small number of variables is the simplex method or one of its variant (Revelle *et al*. 2004). However, the RC beam design is usually neither a linear nor a convex problem. Regardless of the different problem formulations adopted by various authors, there exists nonlinearity in both the objective function and the constraints of the optimization of RC flexural sections. Hence, nonlinear methods should be explored. Non-linear programming approaches can be divided into three large categories: Enumerative, Deterministic, and Heuristic methods (Garcia *et al*. 2006). The nonlinear methods are further discussed here.

3.1 Enumerative approach

The simplest optimization method is exhaustive enumeration. This method is based on generating and evaluating all combinations of the discrete variables. The overall number of evaluation n_e is:

$$n_e = \prod_1^{n_d} p_i \tag{7}$$

where p_i is the number of possible discrete values for each variable and n_d is the number of discrete variables. The optimal solution is obtained by examining the list of feasible solutions against the objective function. This method is conceptually simple and guarantees the global optimum, but the computational time can be impractically large.

To speed up the location of the global optimum, the search space is represented as a decision tree where nodes represent discrete variables and edges represent possible values for the parent node. With the availability of a bounding function, parts of the search space that do not contain the global minimum can be skipped. The base technique is called branch-and-bound. In the worst case it amounts to an exhaustive search but performs much better in practice. Several variations have been implemented, e.g. branch-and-price, branch-and-cut, and branch-reduce-cut.

Search strategies for (mixed) integer linear programming can be found in (Linderoth and Savelesbergh 1999) and in the reference book (Nemhauser and Wolsey 1999) (see also the more recent book (Chen *et al*. 2010)). A history of integer programming including the latest developments can be found in (Juenger *et al*. 2010), while a very recent survey is available in (Burer and Letchford 2012).

3.2 Deterministic approaches

Deterministic methods use a successive search within the solution space which can be based on the objective function, its gradient information or both. They focus on optimization of continuous variables. The optimization process starts with a possible solution and finds the next iteration either by finding a descent direction and computing a step in that directly (line search strategy) or Table 4 Deterministic optimization techniques adopted by various research studies by solving a simple optimization subproblem (trust region strategy (Nocedal and Stephen Wright 2006)).The search continues until a stopping condition is satisfied, typically until a first order optimality condition, usually the Karush-Kuhn-Tucker (KKT) condition is found within the admissible error. The calculations for search direction can be done through various methods with one of the most

Table 4 Deterministic optimization techniques adopted by various research studies

Study	**Deterministic optimization technique**
Norman (1964)	Simple derivatives (one variable)
Sandhu (1971)	Simple differentiation (one variable at a time)
Friel (1974)	Lagrange multiplier technique
Goble and Moses (1975)	Penalty function programming
Chou (1977)	Lagrange multiplier technique
Balaguru (1980a)	Lagrange multiplier technique
Balaguru (1980b)	Simple differentiation (one variable at a time)
Colin and Macrae (1984)	Solve a sequence of linearly constrained nonlinear programs†
Prakash *et al*. (1988)	Lagrange multiplier technique
Chakrabarty (1992a)	Geometric or standard nonlinear programming algorithms
Chakrabarty (1992b)	Geometric programming technique - the newton-raphson method
Adamu and Karihaloo (1994b)	Discretized continuum-type optimality criteria
Adamu and Karihaloo (1994a)	Discretized continuum-type optimality criteria
Adamu and Karihaloo (1994c)	Augmented lagrangian technique
Chung and Sun (1994)	Sequential linear programming algorithm
Al-salloum and Siddiqi (1995)	Lagrange multiplier technique
Ceranic and Fryer (2000)	Lagrange multiplier technique
Dole *et al*. (2000)	Polynomial optimization technique
Dole *et al*. (2000)	Simple differentiation (one variable)
Guerra and Kiousis (2006)	Sequential quadratic programming algorithm
Narayan and Venkataramana (2007)	Sequential unconstrained minimization technique

†As specified in (Hock and Schittkowski 1983), the code FCDPAK implements (Robinson 1972)

popular methods being the BFGS (Broyden-Fletcher-Goldfarb-Shanno) quasi-Newton method with line search and Wolfe condition that requires the function and the gradient value at each iterate.

Evidently, the adopted technique by different researchers is based on the complexity of the problem formulation (see Table 4). In the simplest case, the design variables are determined by simple derivations as only one parameter (or one parameter at a time) is assumed as variable (Norman 1964, Sandhu 1971, Balaguru 1980b). Goble and Moses (1975), however, used a penalty method to find the optimal dimensions of a RC beam and the area of steel reinforcement, and suggested utilizing Powell's (1964) search method that does not depend on the derivatives of the function (modern interior point methods are more appropriate nowadays). Nevertheless, the most widely used methods for RC beam optimization are the gradient-based optimization techniques, such as the geometric programming technique (Chakrabarty 1992a,b), the sequential linear and quadratic programming algorithm (Chung and Sun 1994, Guerra and Kiousis 2006). In the simpler case, the KKT optimality conditions can be solved and an analytic solution obtained [1]explicitly (Friel 1974, Chou 1977, Balaguru 1980a, Al-salloum and Siddiqi 1995, Ceranic and Fryer 2000) or

[1]Solving the optimality conditions of a constrained optimization problem is sometimes called the Lagrangian technique (or Lagrange multiplier technique) as it requires writing the Lagrangian, computing its gradient, and solving the resulting system.

Table 5 Stochastic optimization techniques adopted by various research studies

Study	**Stochastic optimization technique**
Coello and Hernández (1997)	Genetic algorithm
Rajeev and Krishnamoorthy (1998)	Genetic algorithm
Camp *et al*. (2003)	Genetic algorithm
Lee and Ahn (2003)	Genetic algorithm
Lepš and Šejnoha (2003)	Simulated annealing
González-Vidosa *et al*. (2008)	Augmented simulated annealing
McCluskey and McCarthy (2009)	Particle swarm algorithm
Ozturk *et al*. (2013)	Artificial bee colony (swarm-based)
Medeiros and Kripka (2013)	Simulated annealing

with a computer (Prakash *et al*. 1988).

Gradient-based optimization techniques are more efficient. They usually enjoy a linear rate of convergence for first-order methods and even superlinear rate for second-order methods (Newton and quasi-Newton methods) as opposed to a sub-linear rate for the non-gradient-based optimization methods. However, since gradient-based methods find iterations based on derivative information, they can at best guarantee local optimality. Global optimality can only be achieved for convex problems (Nocedal and S. Wright 2006, Theorem 2.5 p.16).Therefore, depending on the problem formulation, deterministic optimization methods may not be the best approach for the optimal RC beam design.

3.3 Stochastic and heuristic approaches

Stochastic and heuristic methods search for the optimal solution using probability rules and an oriented random manner (Sivanandam and Deepa 2008). When deterministic methods fail to find the global optimum of the objective function, or they are computationally too expensive and time consuming, stochastic methods may be used to provide a satisfactory solution in a timelier manner. Stochastic methods are most suited to problems of higher complexity and higher number of variables and constraints. Therefore, to satisfactorily scan all the regions of the problem domain for the optimal solution, stochastic methods usually require numerous computer calculations. Given the increase in computational power, stochastic methods are attracting increasing popularity among researchers in engineering. The main advantages of these methods over the conventional optimization techniques include the following: they do not require gradient information; constraints do not need to be explicit; and restrictions about the search space (e.g. continuity) do not prevent the application of these methods. Various heuristic optimization methods have been applied in structural engineering (Hare *et al*. 2013). The most commonly adopted techniques in this category are simulated annealing (Salamon *et al*. 2002), evolutionary algorithms (Sivanandam and Deepa 2008) and swarm-based optimization algorithms (Olsson 2011).

Using stochastic methods for design of RC beams are introduced just before the turn of the century by the likes of Coello and Hernández (1997) and Rajeev and Krishnamoorthy (1998) through implementation of genetic algorithms (see Table 5). Genetic algorithm (GA) is inspired by the process of natural evolution (Goldberg 1989). In this approach, a population of individuals (variables) is evolved to produce better solutions. The evolution progresses in generations. In each generation, the fitness of the individuals is evaluated against a fitness function (objective function); better fit individuals are selected and randomly modified to produce the next generation; and iterations continue until a maximum number of generations are formed or a satisfactory level of fitness is achieved.

Coello and Hernández (1997) utilize GAs with various representation schemes such as floating point and binary representation. They assume a continuous search space and they adjust different parameters of the GA to obtain reasonable solutions in shorter times. Rajeev and Krishnamoorthy (1998), however, argue that using continuous design variables requires further modification to the solution obtained and therefore does not represent the realistic constraints and does not provide rational solutions. In their paper, Rajeev and Krishnamoorthy include detailing and other construction related constraints in the optimization problem and use GA-based methodologies to optimize the design of reinforced concrete frames (including beams and columns) with discrete design variables. Camp *et al*. (2003) and Lee and Ahn (2003) also use discrete variables and implement GAs to provide optimization procedures for flexural design of simply supported beams, uniaxial columns and multistory frames.

The simulated annealing algorithm emulates the physical process of crystallization of a melted solid. As the mass cools down slowly, higher energy configuration of crystals reduces and eventually the solid reaches the minimum energy configuration. The iterations in this method are based on probabilistic information and decisions as to stay in a state or to move to a neighbouring state, which should ultimately lead the structure to lower levels of energy. González-Vidosa *et al*. (2008) use simulated annealing procedures to provide optimal solutions to the design of RC walls and frames. They reiterate that the restricting constraints in beam design are those related to flexure, shear and deflection of the beams. Lepš and Šejnoha (2003) employ a version of the augmented simulated annealing method to solve the optimal design of a RC beam with discrete variables, and they account for both flexural and shear reinforcements.

4. Spreadsheet implementation and illustrative example

The review of the literature suggests that the design optimization of RC beams as isolated members can be achieved using deterministic optimization techniques and there is no need to use heuristic methods. Although recent advancement in computational power and consequent popularity of stochastic and heuristic methods can be attractive, the more reliable deterministic approaches can be sufficient for flexural design optimization of RC beams, given the small number

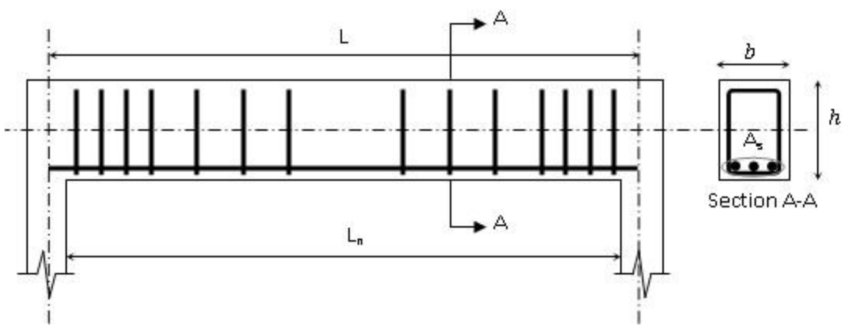
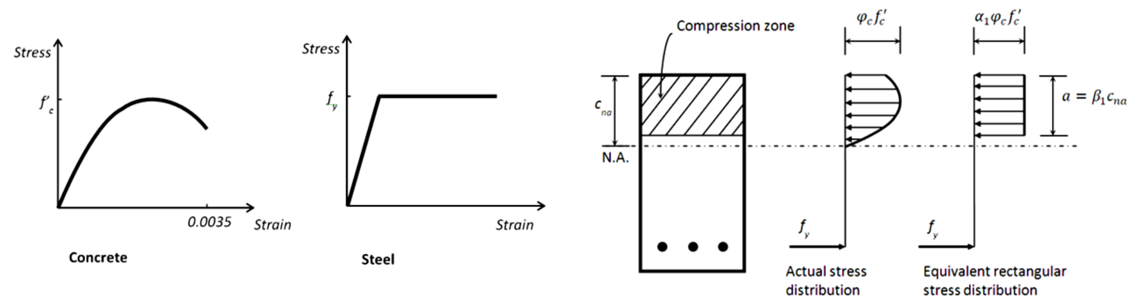


Fig. 2 A singly reinforced concrete beam



(a) Stress-strain diagrams for concrete and steel (b) Actual and equivalent rectangular stress distribution

Fig. 3 Stress distribution in a reinforced concrete beam

of variables. The optimization problem can be implemented in the familiar spreadsheet environment to promote its use by engineers. This section presents such spreadsheet implementation of the RC beam optimization problem and demonstrates, through an example, the efficiency of deterministic methods in finding the optimal solution.

4.1 Problem formulation

4.1.1 Design variables

Assuming the singly reinforced rectangular beam illustrated in Fig. 2, the design variables are the beam width, depth and the area of steel reinforcement that can be optimized by applying the structural and practical constraints. As mentioned earlier in section 2.4, the values that these variables can assume are restricted by practical implementations and are, hence, of a discrete nature. To perform the optimization in a discrete search space, the width and depth of the beam are expressed as a function of a user-defined precision. This dimension precision refers to nominal beam sizes and the practical limit for the accuracy of formwork measurement at the construction site. For example, if dimension precision is set to 50mm, the beam dimensions are rounded to the nearest 50mm. Therefore, each dimension is expressed as the product of an integer multiplier times the dimension precision (see Eqs. (8)-(9)). Therefore, the discrete decision variables to be used in optimization are the integer multipliers for width and depth of the beam (bpm and hpm in Eqs. (8)-(9)).

$$b = bpm * precision \tag{8}$$

$$b = bpm * precision \tag{9}$$

The area of the reinforcement bars is also a function of the area of each bar. The size of the bars used is defined by the user; hence, the area of steel can be expressed as:

$$As = N_b * area\ of\ each\ tensile\ bar \tag{10}$$

4.1.2 Objective function

The objective of the optimization is to *minimize* the overall cost of the beam including both material and construction cost. The objective function is expressed in Eq. (11)

$$f(b, h, A_s) = c_1 A_s + c_2 bh + c_3(2h + b) \tag{11}$$

where

$f(b, h, A_s)$ is the cost per unit length of the beam, and c_1, c_2 and c_3 are constant coefficients that can be defined by the user:

c_1 is the cost coefficient due to the volume of tensile reinforcement steel
c_2 is the cost coefficient due to the volume of concrete
c_3 is the cost coefficient due to shuttering along the surfaces of the beam.

4.1.3 Constraints

The constraints of the optimization problem can be divided into three categories: ULS and SLS constraints prescribed in CSA A23.3-04 (Canadian Standards Association 2004), and other practical constraints resulting from architectural or construction limitations.

Flexural and shear ULSs ensure that flexural stresses in the RC beam do not exceed the material strength. Fig. 3(a) illustrates the material properties of concrete and steel reinforcements. The stress-strain relationship of concrete is a nonlinear curve, while the stress-strain relationship for steel can be represented by an elastoplastic diagram. Using the actual nonlinear stress-strain curve for concrete is not practical for design purposes. Therefore, CSA A23.3-04 allows for an equivalent rectangular stress block to be used instead of the nonlinear stress distribution, as demonstrated in Fig. 3(b).The depth and magnitude of the equivalent stress block are calculated using $\alpha_1$and β_1, mathematical parameters that ensure the compressive stress resultants of the actual and the equivalent rectangular stress distribution are equal. Therefore, in quantifying the constraints, the bending resistance of the section is calculated using the equivalent rectangular stress distribution.

Optimization constraints resulting from ULSs can be summarized as:

1. Maximum bending moment resistance constraint
2. Minimum reinforcement constraint
3. Maximum reinforcement constraint
4. Maximum factored shear resistance constraint

The following constraints result from SLSs:

5. Maximum deflection constraint
6. Maximum crack opening constraint

Other practical constraints include:

7. Bar spacing constraint
8. Minimum and maximum beam width constraint
9. Minimum and maximum beam depth constraint
10. Minimum and maximum beam depth to width ratio constraint

The detailed equations for the constraints are presented in the appendix.

4.2 Spreadsheet implementation

Using the aforementioned objective function and constraints, design optimization of the singly reinforced rectangular beam can be implemented in the familiar MS Excel spreadsheet environment. A worksheet is assigned to input all the user-defined parameters that define the material properties, design requirements and practical restrictions, as depicted in Fig. 4. Calculations are accommodated in a separate worksheet and the optimization results are presented in the third worksheet (Fig. 5).

The Solver Add-In embedded in MS Excel can be used to run the optimization. The decision variables and the constraints are added to Solver through a VBA macro assigned to user buttons. Running the solver minimizes the objective function and determines the optimal values for the design variables.

The Solver Add-In in MS Excel 2010 offers three techniques for solving optimization problems: Simplex method for linear problems, Generalized Reduced Gradient (GRG) algorithm (Lasdon and Waren 1978) for optimizing smooth nonlinear problems, and Evolutionary method for non-smooth nonlinear problems which utilized genetic algorithms. The optimal beam design is a nonlinear problem and either the GRG Nonlinear or the Evolutionary options can be used in MS Excel. However, given the non-convex nature of the problem, none of the two techniques guarantee the global optimum solution. In fact, depending on the starting values of the decision variables, Solver may fail to converge to a feasible solution at all.

A good starting point can help the optimization process. Engineering practice suggests estimates for the dimensions of the beam that can be used as starting point. The overall depth of the beam (h) can be estimated about 10% of the beam span to make sure that the deflection criteria is met. The width of the beam (b) can then be calculated using a depth to width ratio of about 1.5. The area of reinforcement (A_s) should be selected so that ductile behaviour is ensured. If the onset of yielding in tensile reinforcement and crushing in concrete occurs simultaneously, the area

User specified material properties			
Concrete			
Compressive strength	f'_c	25	MPa
maximum size of aggregate	a_{max}	20	mm
Unit weight force	w_c	23500	N/m^3
Material cost		2	\$/m^3
Steel			
Yield strength	f_y	400	MPa
Modulus of elasticity	E_s	200000	MPa
Material cost		250	\$/m^3
Nominal diameter of tensile bars		25	mm
Nominal diameter of stirrups		10	mm
Minimum number of tensile bars		2	
Shuttering			
Material and construction cost		1	\$/m^2
Design parameters			
Uniformly Distributed Dead Load	DL	15	kN/m
Uniformly Distributed Live Load	LL	25	kN/m
Concrete cover provided	c	30	mm
Maximum allowable deflection		Ln/240	
Beam dimensions and exposure condition			
Exposure class		N	
Span of the beam	L	5	m
Clear span of the beam	Ln	4.6	m
Dimension precision		50	mm
Beam width			
Minimum		300	mm
Maximum		1000	mm
Beam depth			
Minimum		100	mm
Maximum		800	mm
Depth to width ratio			
Minimum		0.5	
Maximum		3	

Fig. 4 User-defined input worksheet

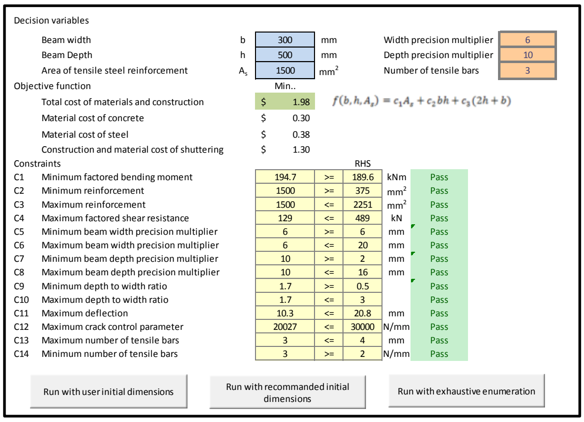


Fig. 5 Optimization worksheet

of reinforcement is at balanced condition. Using about 40% of the reinforcement at balanced condition should result in ductile behaviour. Hence, the initial value of reinforcement can be estimated. Running the optimization with the proposed starting point (usually a feasible solution) can enhance the optimization process. The effect of the starting point is numerically studied in the next section.

4.3 Numerical results

This section presents a numerical example to illustrate the efficiency of various optimization methods in quantifying the optimal design of a RC beam. The assumed RC beam spans 5m, is rectangular in cross-section and is reinforced with a single layer of tensile steel bars (see Fig. 2). The material properties and loading of the beam are presented in Fig. 4.

Using the Evolutionary and GRG Nonlinear options in Solver, the optimization does not always result in a feasible solution, i.e., not all constraints are satisfied. Although using the estimated dimensions and reinforcement as the starting point can help converging to a feasible solution, it does not guarantee such results. To compare the optimization results with the absolute global optimum solution, a VBA code is developed that enumerates all the possible combinations of the decision variables and compares the feasible solutions to determine the global optimal design.
The degree of convergence in Solver is set to 0.001 for both GRG Nonlinear and Evolutionary methods. The population size in Evolutionary method is set to 100 and the mutation rate is 0.075. The estimated starting point for this example is $b = 350\,mm, h = 500\,mm, A_s = 1500\,mm^2$. Table 6 presents the result of the three optimization techniques and their processing times.

According to the exhaustive enumeration, the number of all possible combinations of decision variables is 3,150 but only 874 combinations are feasible for this typical RC beam design and with such a small search space, the global optimum can be found in less than a second.

The GRG Nonlinear method fails to provide a feasible solution with the recommended starting point; however, if the optimization is run for a second time using the result of the first run as the starting point, the program returns the absolute global optimum. The optimization takes less than a second to complete with a typical office computer. On the other hand, the Evolutionary method takes about 36 seconds to complete and still fails to move from the starting point (which is a

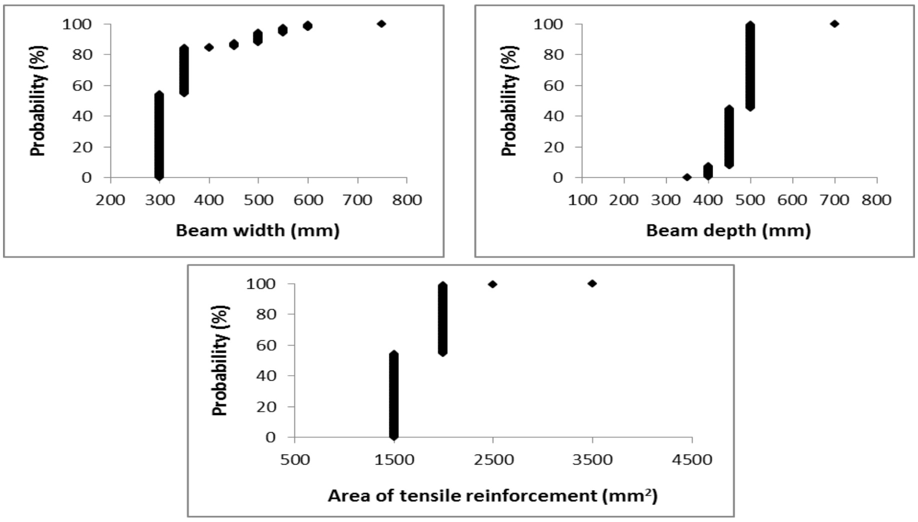


Fig.6 Probability distribution of feasible solutions from GRG Nonlinear Solver

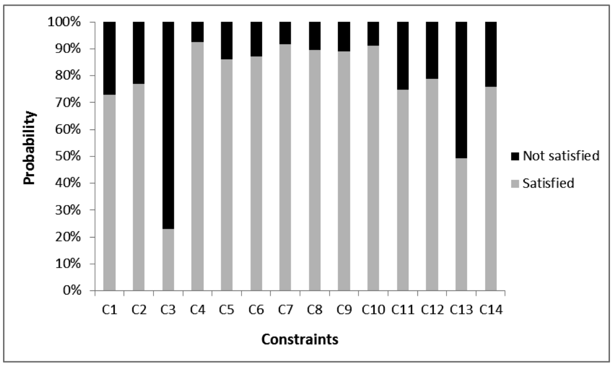


Fig. 7 Probability of optimization constraints being met with the solver GRG Nonlinear method

Table 6 Numerical results for optimal design of a singly RC beam using various optimization methods

Optimization technique	b (mm)	h (mm)	A_s (mm^2)	Total cost (relative)	Feasible	Processing time (sec)
GRG nonlinear	300	400	2500	1.97	No	0.16
Evolutionary	350	550	1500	2.21	Yes	36.64
Exhaustive Enumeration	300	500	1500	1.98	Yes	0.04

feasible solution). Changing the various parameters of the algorithm such as the mutation rate or the population size does not change the result, although requiring bounds on the variables will reduce the processing time to less than a second.

Based on the literature reviewed in previous sections, deterministic nonlinear methods should be sufficient for finding the optimal solution to the problem of RC beam design with relatively limited search space. But is the GRG Nonlinear method of MS Excel suited to solving this problem? To answer this question, the efficiency of the GRG Nonlinear method is examined using various starting points for optimization. A VBA code was implemented to feed the spreadsheet with all the possible combinations of design variables (based on upper bound and lower bound constraints limiting the width and depth of the beam and the number of reinforcing bars) and to calculate the results based on the GRG Nonlinear Solver. The optimization results are highly dependent on the starting points. Using the input parameters defined in Fig. 4, there are a total of 3150 possible combinations of design variable; i.e., 3150 possible starting points for the optimization. The GRG Nonlinear Solver returns a feasible solution with only 8% of the starting points, but 55% of these feasible solutions are equal to the global optimum. In other words, although the Solver rarely comes to a feasible solution, if a solution is found, there is a good chance of it being the global optimum. Fig. 6 shows the probability distribution of the dimensions and reinforcement of the RC beam as returned by the GRG Nonlinear Solver when feasible solutions are returned. It is evident that the Solver feasible results are generally very close to the global optimum values provided a feasible solution is obtained.

A closer examination of GRG Nonlinear Solver results with various starting points shows that optimization fails to meet the maximum reinforcement constraint with 77% of the starting points. This constraint is a critical one that ensures ductile failure of the beam. Fig. 7 demonstrates the probability of failure of all the optimization constraints (notations for constraints refer to those introduced in Fig. 5). Constraint C13, the maximum number of bars, is next with 51% chance of not being satisfied. This constraint refers to the physical possibility of arranging tensile bars in one row as defined earlier. Constraints C11 and C12 correspond to serviceability limit states (SLS) of deflection and cracking. These constraints are usually governing design factors for long-span beams. Hence, with the aim of improving the optimization, it is reasonable to relax these constraints from optimization and later check the SLS on the optimal solution. The analysis is repeated with SLS constraints relaxed, however, the results hardly improve. The rate of feasibility of GRG Nonlinear solutions is slightly raised to 11%, and the probability of obtaining the global optimum is 52% of the feasible solutions. The probability distribution of design variables and failure rates of various constraints are also very similar.

In conclusion, the feasibility rate of the solutions obtained from the GRG Nonlinear method is low, while optimality of the feasible solutions is high. The overall results suggest that although running the nonlinear optimization method available in MS Excel is quick, it is highly dependent on the starting point and not efficiently reliable. In contrast, the exhaustive enumeration technique can be efficiently implemented with VBA to provide the optimal design and ensure global optimality.

4.4 Sensitivity analysis

To investigate the effect of various input parameters on the overall cost of a RC beam, a sensitivity analysis is performed using the developed spreadsheet. The main variable input parameters considered are the strength and cost of concrete and steel reinforcement and also the size of reinforcing bars. The bending moment demand on the beam is usually the governing factor in designing the size and reinforcement of typical RC beam sections. Therefore, to perform the sensitivity study, three levels of applied bending moment have been considered: Low, medium and high. This is achieved by keeping a constant span and increasing the applied dead and live loads on the input worksheet shown in Fig. 4.

A common sensitivity analysis available in MS Excel is the Pearson correlation method.

Pearson correlation coefficient measures the extent of linear correlation between normally distributed variables (Veaux *et al*. 2012). However, when variables are not normally distributed or nonlinear correlations are examined, Spearman's rank correlation coefficient provides a moreappropriate method by assessing the degree of monotonic (not necessary linear) correlation between two variables. In this study, the input parameters are uniformly distributed and the correlation between the input parameters and the overall cost of construction is not necessarily linear. Hence, the Spearman's rank correlation coefficient is applied for sensitivity analysis.

Table 7 shows the range of the five input parameters that are varied uniformly to produce the data. A VBA code is developed to feed the spreadsheet with all the combinations of input variables and calculate the respective cost of construction. The procedure is repeated for three levels of applied bending moment. The data is then ranked to facilitate the calculation of Spearman's rank correlation coefficient. Fig. 8 shows the results in the form of tornado graphs for the three bending moment demand levels (low: 187kNm, medium: 458kNm and high: 743kNm). For all three bending moment levels, the input parameter that affects the overall cost is the cost of concrete. The rank correlation coefficient for concrete cost is 0.52 for low bending moment, which increase to

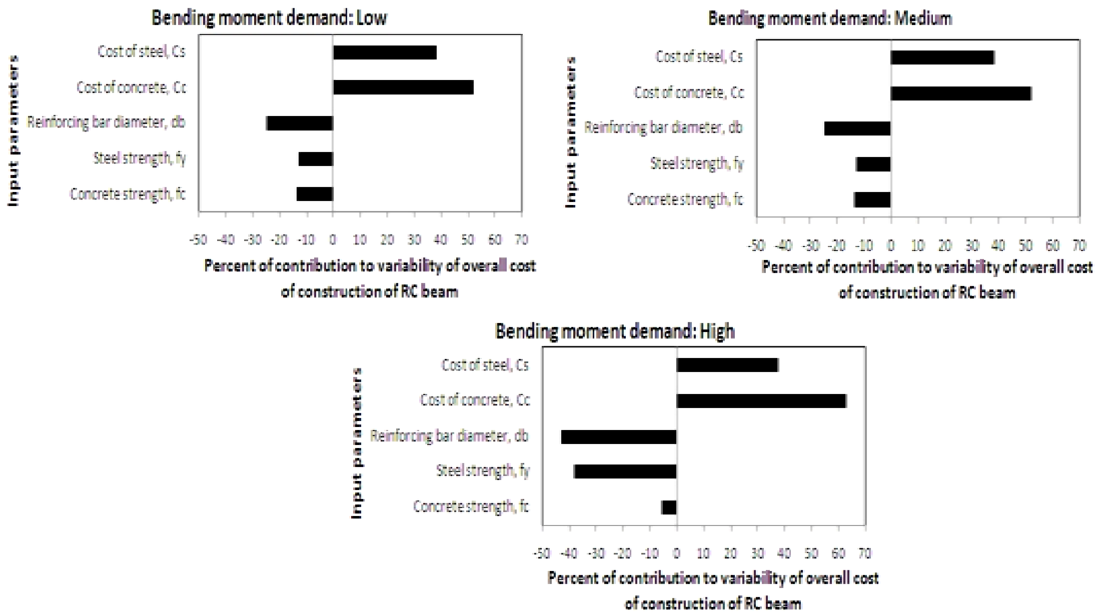


Fig. 8 Percent of contribution of various input parameters to the variability of the overall cost of construction for three levels of bending moment demand

Table 7 Range of uniform distribution of input parameters used for sensitivity analysis

Input parameters	Concrete strength f'_c (MPa)	Steel strength f_y (MPa)	Reinforcing bar diameter d_b (mm)	Cost of concrete C_c (relative to cost of formworks)	Cost of steel C_s (relative to cost of formworks)
Range	25 - 40	250 - 400	10 -55	1 - 5	100 - 300

0.58 and 0.63 for medium and high bending moment levels, respectively. The positive value of the coefficients means higher values of cost of concrete result in higher value of the overall cost.

When bending moment demand is high, the diameter of the reinforcing bars is the next most influential input parameter on the overall cost of construction of RC beams. The negative value of the Spearman's rank correlation coefficient confirms the higher the diameter of reinforcing bars is, the lower the cost of construction will be. With low and medium bending moment levels, higher strength of steel and concrete only slightly reduce the overall cost, with a correlation just over 10%. However, when bending moment level is high, strength of steel plays a significant role in reducing the overall cost, with a correlation close to 40%.

In conclusion, as can also be predicted by inspection, the sensitivity analysis confirms that reducing the cost of concrete and steel can significantly decrease the overall cost of construction of RC beam. The strength of material within the common range has the minimum effect on the overall cost, as long as bending moment demand is considered low or medium. However, with high bending moment demand, choosing higher strength steel and larger bar sizes in design of RC beams can substantially reduce the overall cost of construction.

5. Conclusions

This paper presents a review of the available literature on the design optimization of reinforced concrete beams as structural members. A comprehensive optimization of a structure ideally studies the structure as one entity and takes into account the cost of materials, construction and maintenance, as well as functional and structural constraints. However, given the complex nature of many concrete structures, it is common to optimize the design of individual components of a structure to achieve a more economical design for the whole structure. The optimal design of concrete beams, either individually or as part of a frame, has been addressed by many research studies using various optimization approaches depending on the problem formulation. The objective of optimization (e.g. minimum cost, weight ...), the design variables and the constraints considered by different studies vary widely and hence, different optimization methods have been employed to provide the optimal design. The review suggests that cost optimization of a concrete beam is a discrete, but nonlinear and non-convex problem in nature, yet it can be achieved using deterministic approaches rather than heuristic ones, especially when the beam is considered as an isolated member and the number of design variables is limited.

This paper also presents a spreadsheet implementation of the optimization of concrete beams and a numerical example to demonstrate the efficiency of deterministic methods. The results indicate that given the small search space of many typical RC beam designs and the instability of nonlinear approaches and their dependency on the starting point, exhaustive enumeration is the most efficient and reliable method that guarantees global optimality. Exhaustive enumeration method is comfortably implemented in the MS Excel spreadsheet using VBA to promote its use by engineers.

Using the enumerative method implemented, a sensitivity analysis is performed to evaluate the influence of different input parameters on the optimal cost of construction. The results suggest that cost of concrete and steel expectedly have the highest influence on the overall cost of construction of RC beam; however, when bending moment demand is high on the beam, using higher strength steel and larger reinforcing bar sizes can greatly reduce the overall costs.

Acknowledgements

The first author was supported by an Engage grant from the Natural Sciences and Engineering Research Council of Canada (NSERC). The second author acknowledges support from NSERC through the Discovery grant program, and the Canadian Foundation for Innovation (CFI) through the Leaders Opportunity Fund. The last author was supported by an NSERC Discovery grant.

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Appendix

Optimization constraints for a singly reinforced rectangular beam can be expressed as the following.

1. Bending moment resistance: $M_f + M_{sw} \leq M_r$

M_f	Factored design bending moment (constant user-defined value)
M_{sw}	Bending moment due to self-weight of the beam: $M_{sw} = \frac{Dw_c bhL^2}{8}$
w_c	Unit weight force of concrete (constant user-defined value)
D	Load factor for dead load (constant code parameter)
L	Span of beam (constant user-defined value)
M_r	The factored bending moment resistance: $M_r = \varphi_s f_y A_s \left(d - \frac{a}{2}\right)$
φ_s	Resistance factor for steel (constant code parameter)
f_y	Yield strength of steel reinforcement (constant user-defined value)
d	Effective depth of beam: $d = h - c - \frac{d_b}{2} - d_s$
c	Concrete cover provided (constant user-defined value)
d_b	Diameter of the bending reinforcement bars (constant user-defined value)
d_s	Depth of the equivalent rectangular stress block: $a = \varphi_s f_y A_s / \alpha_1 \varphi_c f_c' b$
a	Depth of the equivalent rectangular stress block: $a = \varphi_s f_y A_s / \alpha_1 \varphi_c f_c' b$
f_c'	Compressive strength of concrete (constant user-defined value)
φ_c	Resistance factor for concrete (constant code parameter)
α_1, β_1	Equivalent rectangular stress distribution parameters (code prescribed constant based on the compressive strength of concrete)

2. Minimum reinforcement: $A_s \geq \frac{0.2\sqrt{f_c'}}{f_y} bh$

3. Maximum reinforcement: $A_s \leq \left(\frac{0.75 \times 700}{700 + f_y}\right)\left(\frac{\alpha_1 \beta_1 \varphi_c f_c'}{\varphi_s f_y}\right) bd$

4. Maximum factored shear resistance: $0.25 \varphi_c f'_c b d_v \geq V_f$

V_f	Factored design shear (constant user-defined value)
d_v	Effective shear depth: $Max[0.9d, 0.72h]$

5. Maximum deflection: $\Delta_i \leq \Delta_{max}$

Δ_{max}	Maximum allowable deflection (code prescribed constant based on beam span)
Δ_i	Immediate deflection: $\Delta_i = \left(\frac{5}{48}\right)\frac{M_a L^2}{E_c I_e}$
M_a	Unfactored design moment (constant user-defined value)
E_c	Modulus of elasticity of concrete (calculated constant): $E_c = 4500\sqrt{f'_c}$
I_e	Effective moment of inertia: $I_e = I_{cr} + (I_g - I_{cr})\left(\frac{M_{cr}}{M_a}\right)^3$
M_{cr}	Cracking moment: $M_{cr} = \frac{2 f_r I_g}{h}$
f_r	Modulus of rupture: $f_r = 0.6\sqrt{f'_c}$
I_g	Gross moment of inertia: $I_g = \frac{bh^3}{12}$

I_{cr} Moment of inertia of the cracked section: $I_{cr} = \frac{b\bar{y}^3}{3} + nA_s(d-\bar{y})^2$

n Modular ratio: $n = \frac{E_s}{E_c}$

E_s Modulus of elasticity of steel (constant user-defined value)

$\bar{y}$ Neutral axis depth of cracked section: $\bar{y} = \frac{-nA_s + \sqrt{(nA_s)^2 + 2bd(nA_s)}}{b}$

6. Maximum crack control parameter: $z \le z_u$

z_u Upper limit for crack control parameter (constant code parameter)

z Crack control parameter: $z = f_s \sqrt[3]{d_c A}$

d_c The distance from the extreme tension fibre to the centre of the longitudinal bar located closest thereto: $d_c = Min\,[c, 50mm] + \frac{d_b}{2} + d_s$

A Effective tension area of concrete surrounding the flexural reinforcement: $A = \frac{A_e}{N_b}$

A_e Total effective tension area: $A_e = b(2d_c)$ for one row of tensile bars

f_s Stress in steel at maximum service load: $f_s = 0.6f_y$

7. Bar spacing constraint: $N_l \le N_b \le N_u$

N_b Number of bending reinforcement bars

N_l Lower limit for number of reinforcement bars (constant user-defined value)

N_u Upper limit for number of bending reinforcement bars: $N_u = Int\left(\frac{b + S_{min} - 2c - 2d_s}{d_b + S_{min}}\right)$

S_{min} Minimum clear bar spacing: $S_{min} = Max[1.4d_b, 1.4a_{max}, 30mm]$

a_{max} Maximum aggregate size (constant input)

8. Beam width constraint: $b_l \le b \le b_u$

b_l, b_u Lower and upper limits for beam width (constant user-defined values)

9. Beam depth constraint: $h_l \le h \le h_u$

h_l, h_u Lower and upper limits for beam depth (constant user-defined values)

10. Beam depth to width ratio constraint: $R_l \le h/b \le R_u$

R_l, R_u Lower and upper limits for beam depth to width ratio (constant user-defined values)

CC