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Free vibrations analysis of arbitrary three-dimensionally FGM nanoplates

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Abstract. In this paper, the free vibrations analysis of the nanoplates made of three-directional functionally graded material (TDFGM) with small scale effects is presented. To study the small-scale effects on natural frequency, modified strain gradient theory (MSGT) has been used. Material properties of the nanoplate follow an arbitrary function that changes in three directions along the length, width and thickness of the plate. The equilibrium equations and boundary conditions of nanoplate are obtained using the Hamilton's principle. The generalized differential quadrature method (GDQM) is used to solve the governing equations and different boundary conditions for obtaining the natural frequency of nanoplate made of three-directional functionally graded material. The present model can be transformed into a couple stress plate model or a classic plate model if two or all parameters of the length scales set to zero. Finally, numerical results are presented to study the small-scale effect and heterogeneity constants and the aspect ratio with different boundary conditions on the free vibrations of nanoplates. To the best of the researchers' knowledge, in the literature, there is no study carried out into MSGT for free vibration analysis of FGM nanoplate with arbitrary functions.

Keywords: vibration; nanoplate; three-directional functionally graded material (TDFGM); modified strain gradient theory (MSGT); generalized differential quadrature method (GDQM); size effect

1. Introduction

Nano is a new scale in technology and a new approach across all scientific fields that empowers human to design and construct by controlling shape and size of nanostructures (Luttge 2016). Nanotechnology has widespread applications in various sciences such as, medical applications, micro and nano electromechanical systems (MEMS & NEMS) and etc. (Ramsden 2016, Javidi *et al*. 2019). Nanostructures are superior in mechanical, chemical and electronic properties due to their small scale effects (Zhang *et al*. 2004; Alizada and Sofiyev 2011, Abouelregal and Zenkour 2019, Ebrahimi and Dabbagh 2019, Rohani Rad and Farajpour 2019). Nanoplates are widely used in nanostructures, such as vibration meters, watches, sensors and micro/nano electromechanical systems. Due to this widespread application, understanding the vibrational properties of nanoplates is an important problem and has been considered in recent studies.

At the nanoplate, the size effects play a significant role. Both experimental and molecular dynamics simulation results show that the small-scale effect cannot be ignored in analyzing the mechanical properties of nanostructures and classical continuum theory cannot be applied (Akgöz and Civalek 2013a, b, Malekzadeh *et al*. 2014, Zeighampour and Tadi Beni 2014a, b, Kolahchi *et al*. 2015, 2017, Li and Hu 2015, 2016a, b, 2017a, b, Li *et al*. 2015, 2016, Ebrahimi *et al*. 2016, Ebrahimi and Barati 2017, Ebrahimi and Dabbagh 2017a, b, Ghorbani Shenas *et al*. 2017, Li *et al*. 2017, Li and Luo 2017, She *et al*. 2017, 2018a, b, c, d, Ajri and Fakhrabadi 2018, Azimi *et al*. 2018, Farajpour *et al*. 2018, Karami *et al*. 2018, Norouzzadeh and Ansari 2018, Preethi *et al*. 2018, Zargaripoor and Bahrami 2018, Zargaripoor *et al*. 2018, Zeighampour *et al*. 2018, Abouelregal 2019, Mohammadi *et al*. 2019, Zarezadeh *et al*. 2019). Molecular dynamics simulation is a suitable method for simulating the mechanical behavior of small-sized structures, but for structures with a large number of atoms is time consuming computationally (Nejad and Hadi 2016b, Farajpour *et al*. 2018). In order to determine the effects of the size of some non-classical theories, such as Eringen's nonlocal theory (Eringen 1983), Couple stress theory (CST) (Yang *et al*. 2002), and Strain gradient theory (SGT) (Mindlin 1965), have been introduced and used in micro and nano structures. The nonlocal theory presented by Eringen was used by Wang *et al*. (2016a) to study the effects of size on a micro scale. However, it has been shown that Eringen's Nonlocal theory predicts a softening effect that is inconsistent with experimental results for some of materials that predict the hardening effect (Abazari *et al*. 2015). Considering the first and second derivatives of the strain tensor, Mindlin proposed a higher order gradient theory for elastic materials. According to Mindlin's formulation, Lam *et al*. (2003) presented the modified SGT by introducing a new equilibrium equation that explains the behavior of higher order stress and couple stress. The SGT is widely used in the analysis of nanostructures (Hosseini *et al*. 2016, 2017, 2018, Adeli *et al*. 2017, Shishesaz *et al*. 2017). Kong *et al*. (2009) investigated the static and dynamic responses of Euler– Bernoulli microbeams using strain gradient elasticity theory. They studied the effect of

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thickness to the material length scale parameter ratio of the microbeams on their static deformation and vibrational behavior. Wang *et al*. (2010) presented Timoshenko microbeams formulations based on the strain gradient elasticity theory. Lazopoulos (2004) employed the Kirchhoff plate theory and investigated the size effect on the bending of strain gradient elastic thin plates. Employing a variational method, Papargyri-Beskou *et al*. (2010) investigated the gradient elastic flexural Kirchhoff plate subjected to static loading and obtained the related boundary conditions. Li *et al*. (2014), applying the strain gradient elasticity theory for the thin layer micro-plate model theory and studied the bending problem of simply micro-plate using the proposed model. Modeling of the sinusoidal plate using the mentioned theory. The FGM circular and annular micro-plate was investigated using the first order shear deformation theory and the generalized differential quadrature method by Ansari *et al*. (2015). Lei *et al*. (2016) studied the size-dependent vibration of nickle cantilevere microbeams based on the strain gradient elasticity theory and by using the differential quadrature method and the least square method, the experimental results were interpreted and the material length scale parameters in the scale of micron in elastic range were obtained. Shenas *et al*. (2017) investigated the thermal buckling of rotating pre-twisted FG microbeams with temperature-dependent material properties based on the strain gradient elasticity theory in conjunction with the first-order shear deformation theory of beams. In the other work, Ansari *et al*. (2016) extracted a Mindlin microplate model based on the strain gradient elasticity theory for bending, buckling and free vibration of FG plate with circular annular geometry. They utilized the differential quadrature method to solve the governing equations. Shenas and Malekzadeh (2016) applied the Chebyshev-Ritz method to consider free vibration of FG quadrilateral microplates with arbitrary boundary conditions in thermal environment. Mirsalehi *et al*. (2017) carried out numerical solutions for buckling and free vibration problems of FG microplates using the spline finite strip method. Thai *et al*. (2017) investigated the different mechanical response of third-order FG microplates using an isogeometric analysis (IGA) method. Hadi *et al*. (2018c) investigated numerical modelling of a spheroid living cell membrane under hydrostatic pressure. The outer surface cell membrane is exposed to hydrostatic pressure and the inner surface is in contact with the cytoplasm. Given that the cell membrane is a spherical shell having a thickness in the nanometer scale; SGT is applied to capture size-dependent behaviors of the cell membrane.

FGMs are inhomogeneous materials, consisting of two (or more) different materials, engineered to have a continuously varying spatial composition profile. The volume fraction of the constituent material is gradually changed, that these changes eliminate inter-plane problems and cause the smoothness of the stress distribution profile (Udupa *et al*. 2014). Over the past decade, FGM has been widely used in various engineering fields such as aerospace, automotive, optical, biomechanics, electronics, chemical, mechanical and shipbuilding industries (Cho and Oden 2000, Haciyev *et al*. 2018, 2019, Seyyed Nosrati *et al*. 2019, Zarezadeh *et al*. 2019). With the development of material technology, FG materials have been used in micro/nano electromechanical systems. Due to the high sensitivity of MEMS and NEMS to external stimuli, understanding their behavior and mechanical properties for the design and construction of MEMS and NEMS made from FG materials is very important (Hosseini-Hashemi and Nazemnezhad 2013). A number of articles have been published in recent years regarding different states of FGM (Ghannad *et al*. 2009, Nejad and Rahimi 2009, Nejad *et al*. 2009, 2014a, b, 2015a, b, 2017a, b, d, 2018a, Ghannad and Nejad 2010, Nejad and Rahimi 2010, Ghannad *et al*. 2012, 2013, Huang *et al*. 2013, Fatehi and Nejad 2014, Nejad and Kashkoli 2014, Ziegler and Kraft 2014, Jabbari *et al*. 2015, 2016, Nejad and Fatehi 2015, Dehghan *et al*. 2016, Mazarei *et al*. 2016, Tadi Beni 2016, Afshin *et al*. 2017, Gharibi *et al*. 2017, Kashkoli *et al*. 2017, Kashkoli and Nejad 2018, Kashkoli *et al*. 2018, Mehralian and Tadi Beni 2018). It should be noted that most analyzes mentioned are related to FGMs, in which material properties change in one direction. However, there are applications that require grading macroscopic properties in two or three directions. As reported by Steinberg (1986), the body of an aerospace device, when it performs at Mach 8 and at a height of 29 kilometers, sustains a very high temperature field with an excessive gradient on the surface and thickness. In these conditions, conventional unidirectional FGMs are not appropriate enough to resistance against multi-directional temperature variations. Therefore, it is important for developing new FGMs with macroscopic properties that change in two or three directions. A large number of papers have been published in the field of two directional FGM nano-beams (Lü *et al*. 2008, Wang *et al*. 2011, Thai and Choi 2013, Şimşek 2015, Deng and Cheng 2016, Zhao *et al*. 2016, Pydah and Batra 2017, Pydah and Sabale 2017, Van Do *et al*. 2017, Nguyen and Lee 2018, Rajasekaran and Khaniki 2018). Lü *et al*. (2009) presented a state space based on the generalized differential quadrature method for thermoelastic analysis of bi-directional FGMs. In addition, the dynamical behavior of a circular plate made of bi-directional FGMs was investigated by Nie and Zhong (2010). Şimşek (2012) presents nonlocal effects in the free longitudinal vibration of axially functionally graded tapered nanorods. Free vibration analysis of axially functionally graded nanobeam with radius varies along the length based on strain gradient theory are investigated by Zeighampour and Tadi Beni (2015). Wang *et al*. (2016b) studied the free vibration of the beam made of bi-directional FGMs, which the material properties change along the length and thickness. They assumed that material properties along the length were based on the simple power distribution with the arbitrary power index and exponential variations along the thickness. Nejad *et al*. (2016) analyzed the buckling of the Bernoulli Euler nanobeam made of bi-directional FGMs using Eringen's nonlocal theory. Nejad and Hadi (2016b) investigated the free vibration of the Bernoulli Euler nanobeam made of bi-directional FGMs using Eringen's nonlocal theory. Also, this authors (Nejad and Hadi 2016a) investigated bending analysis of bi-directional functionally graded Euler–Bernoulli nano-beams use Eringen's non-local elasticity theory. Nejad *et al*. (2017c) applied consistent couple-stress theory for free vibration analysis of Euler-Bernoulli nano-beams made of arbitrary bi-directional functionally graded materials. Nejad *et al*. (2018b) presented bending analysis of bi-directional functionally graded Euler-Bernoulli nano-beams using integral form of Eringen's non-local elasticity theory. Hadi *et al*. (2018a) analyzed vibrations of three-dimensionally graded nanobeams. Hadi *et al*. (2018b) studied buckling of FGM euler-bernoulli nano-beams with 3D-varying properties based on consistent couple-stress theory. Nonlinear bending, buckling and vibration of bi-directional functionally graded nanobeams are presented by Yang *et al*. (2018). Trinh *et al*. (2018) reported size-dependent vibration of bi-directional functionally graded microbeams with arbitrary boundary conditions. Karami *et al*. (2019) presented the resonance behavior of Kirchhoff nanoplate made of 3D-FGMs using the higher-order model of nonlocal strain gradient theory.

According to previous researches, no study has been done on free vibration of nanoplates made of three-directional FGMs using MSGT, so far. In this paper, material properties of plate are arbitrarily function in three directions of thickness, width and length of plate. In the end, the effects of size, power index, and different boundary conditions on free vibrations of nanoplate are shown.

2. Analysis

A nano-plate with the length *a*, width *b* and thickness *h*, made of three-directional FGM, is shown in Fig. 1. The Cartesian coordinates (*x*, *y*, *z*) are considered.

The modulus of elasticity *E* and density *ρ* are considered as an arbitrary symmetric function that changes in three directions of length, width and thickness.

$$E = f_1(x)h_1(y)g_1(z) \tag{1}$$

$$\rho = f_2(x)h_2(y)g_2(z) \tag{2}$$

In accordance with the Kirchhoff plate theory for thin plates, the displacement field is considered as follows (Reddy 2006)

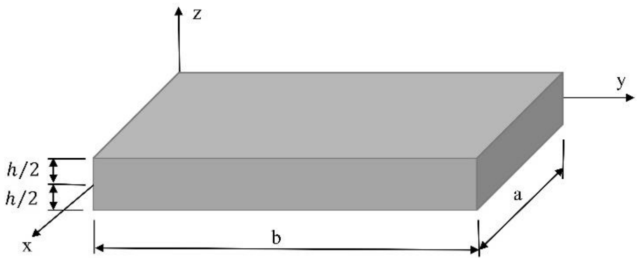


Fig. 1 The schematic of nanoplate made of a three-directional FGM

$$\begin{cases} u(x,y,z,t) = u_0(x,y,t) - z\dfrac{\partial w_0}{\partial x} \\ v(x,y,z,t) = v_0(x,y,t) - z\dfrac{\partial w_0}{\partial x} \\ w(x,y,z,t) = w_0(x,y,t) \end{cases} \tag{3}$$

which u, v, w are displacements in the directions of x, y, z respectively, and u_0, v_0 represent the values of u and v on the middle plane of $(x, y, 0)$ respectively. Due to there is no tensile deformation in the middle plane of the plate, hence $u_0 = v_0 = 0$.

$$\begin{cases} u(x,y,z,t) = -z\dfrac{\partial w_0}{\partial x} \\ v(x,y,z,t) = -z\dfrac{\partial w_0}{\partial x} \\ w(x,y,z,t) = w_0(x,y,t) \end{cases} \tag{4}$$

Lam *et al*. (2003) introduced the MSGT, which includes three material length scale parameters associated with the dilatation gradient vector, deviatoric stretch and symmetric rotation gradient tensors in addition to two Lamé constants for linear elastic isotropic materials. The total strain energy based on the MSGT can be expressed as follows

$$U = \frac{1}{2}\int_{\Omega}\left(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau_{ijk}^{(1)}\eta_{ijk}^{(1)} + m_{ij}^{s}\chi_{ij}^{s}\right)dv \tag{5}$$

The components of strain tensor ε_{ij}, the dilatation gradient vector γ_i, the symmetric rotation gradient tensor χ_{ij}^{s} and the deviatoric stretch gradient tensor $\eta_{ijk}^{(1)}$ are defined as follow

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{6}$$

$$\gamma_i = \varepsilon_{mm.i} \tag{7}$$

$$\chi_{ij}^{s} = \frac{1}{2}\left(\theta_{i,j} + \theta_{j,i}\right) \tag{8}$$

$$\begin{aligned}\eta_{ijk}^{(1)} &= \frac{1}{3}\left(\varepsilon_{jk,i} + \varepsilon_{ki,j} + \varepsilon_{ij,k}\right) \\ &-\frac{1}{15}\left[\delta_{jk}\left(\varepsilon_{mm,i} + 2\varepsilon_{mi,m}\right) + \delta_{ki}\left(\varepsilon_{mm,j} + 2\varepsilon_{mj,m}\right)\right. \\ &\left.+\delta_{ij}\left(\varepsilon_{mm,k} + 2\varepsilon_{mk,m}\right)\right]\end{aligned} \tag{9}$$

$$\theta = \frac{1}{2}\left[curl(\vec{u})\right]_i \tag{10}$$

which u_i and θ_i are the components of displacement vector and rotation vector respectively. For the linear elastic material, the components of stress and higher orders stress are defined as follows

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{xz}\end{Bmatrix} = \frac{E(x,z)}{1-\nu^2}\begin{bmatrix}1 & \nu & 0 & 0 & 0\\ \upsilon & 1 & 0 & 0 & 0\\ 0 & 0 & \dfrac{1-\nu}{2} & 0 & 0\\ 0 & 0 & 0 & \dfrac{1-\nu}{2} & 0\\ 0 & 0 & 0 & 0 & \dfrac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} \tag{11}$$

$$p_i = 2\mu l_0^2 \gamma_i \tag{12}$$

$$\chi_{ij}^s = \frac{1}{2}\left(\theta_{i,j} + \theta_{j,i}\right) \tag{13}$$

$$\tau_{ijk}^{(1)} = 2\mu l_1^2 \eta_{ijk}^{(1)} \tag{14}$$

$$m_{ij}^s = 2\mu l_2^2 \chi_{ij}^s \tag{15}$$

p_i, $\tau_{ijk}^{(1)}$, m_{ij}^s, are the higher order stresses. μ, E and ν are shear modulus, Young modulus, and Poisson ratio l_0, l_1, l_2 in higher order stresses, are the length scale parameters.

Substituting Eq. (4) into Eqs. (6)-(10), the strain tensor and the gradient strain tensor are obtained as follows

$$\theta = \frac{\partial w_0}{\partial y} e_x - \frac{\partial w_0}{\partial x} e_y \tag{16}$$

$$\varepsilon_{xx} = -z\frac{\partial^2 w_0}{\partial x^2}, \quad \varepsilon_{yy} = -z\frac{\partial^2 w_0}{\partial y^2}, \quad \varepsilon_{xy} = -z\frac{\partial^2 w_0}{\partial x \partial y} \tag{17}$$

$$\begin{gathered} \chi_{xx}^s = \frac{\partial^2 w_0}{\partial x \partial y}, \qquad \chi_{yy}^s = -\frac{\partial^2 w_0}{\partial x \partial y}, \\ \chi_{xy}^s = \frac{1}{2}\left(\frac{\partial^2 w_0}{\partial y^2} - \frac{\partial^2 w_0}{\partial x^2}\right) \end{gathered} \tag{18}$$

$$\begin{gathered} \gamma_x = -z\left(\frac{\partial^3 w_0}{\partial x^3} + \frac{\partial^3 w_0}{\partial x \partial y^2}\right), \\ \gamma_y = -z\left(\frac{\partial^3 w_0}{\partial y^3} + \frac{\partial^3 w_0}{\partial x^2 \partial y}\right), \\ \gamma_z = -\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right) \end{gathered} \tag{19}$$

and

$$\eta_{xxx}^{(1)} = \frac{1}{5} z\left(3\frac{\partial^3 w_0}{\partial x \partial y^2} - 2\frac{\partial^3 w_0}{\partial x^3}\right) \tag{20a}$$

$$\eta_{yyy}^{(1)} = \frac{1}{5} z\left(3\frac{\partial^3 w_0}{\partial x^2 \partial y} - 2\frac{\partial^3 w_0}{\partial y^3}\right) \tag{20b}$$

$$\eta_{zzz}^{(1)} = \frac{1}{5}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right) \tag{20c}$$

$$\eta_{xxy}^{(1)} = \eta_{xyx}^{(1)} = \eta_{yxx}^{(1)} = \frac{1}{5} z\left(\frac{\partial^3 w_0}{\partial y^3} - 4\frac{\partial^3 w_0}{\partial x^2 \partial y}\right) \tag{20d}$$

$$\eta_{xxz}^{(1)} = \eta_{xzx}^{(1)} = \eta_{zxx}^{(1)} = \frac{1}{15}\left(\frac{\partial^2 w_0}{\partial y^2} - 4\frac{\partial^2 w_0}{\partial x^2}\right) \tag{20e}$$

$$\eta_{yyx}^{(1)} = \eta_{yxy}^{(1)} = \eta_{xyy}^{(1)} = \frac{1}{5} z\left(\frac{\partial^3 w_0}{\partial x^3} - 4\frac{\partial^3 w_0}{\partial x \partial y^2}\right) \tag{20f}$$

$$\eta_{yyz}^{(1)} = \eta_{yzy}^{(1)} = \eta_{zyy}^{(1)} = \frac{1}{15}\left(\frac{\partial^2 w_0}{\partial x^2} - 4\frac{\partial^2 w_0}{\partial y^2}\right) \tag{20g}$$

$$\eta_{zzx}^{(1)} = \eta_{zxz}^{(1)} = \eta_{xzz}^{(1)} = \frac{1}{5} z\left(\frac{\partial^3 w_0}{\partial x^3} + \frac{\partial^3 w_0}{\partial x \partial y^2}\right) \tag{20h}$$

$$\eta_{zzy}^{(1)} = \eta_{zyz}^{(1)} = \eta_{yzz}^{(1)} = \frac{1}{5} z\left(\frac{\partial^3 w_0}{\partial y^3} + \frac{\partial^3 w_0}{\partial x^2 \partial y}\right) \tag{20i}$$

$$\eta_{xyz}^{(1)} = \eta_{yzx}^{(1)} = \eta_{zxy}^{(1)} = \eta_{xzy}^{(1)} = \eta_{yxz}^{(1)} = \eta_{zyx}^{(1)} = -\frac{1}{3}\frac{\partial^2 w_0}{\partial x \partial y} \tag{20j}$$

By substituting the Eqs. (16)-(20) into Eq. (5), the total strain energy is written as follows

$$\begin{aligned} U = \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^b\int_0^a \Bigg[& -\sigma_{xx} z\frac{\partial^2 w_0}{\partial x^2} - \sigma_{yy} z\frac{\partial^2 w_0}{\partial y^2} \\ & -2\sigma_{xy} z\frac{\partial^2 w_0}{\partial x \partial y} - p_x z\left(\frac{\partial^3 w_0}{\partial x^3} + \frac{\partial^3 w_0}{\partial x \partial y^2}\right) \\ & -p_y z\left(\frac{\partial^3 w_0}{\partial y^3} + \frac{\partial^3 w_0}{\partial x^2 \partial y}\right) - p_z\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right) \\ & +\tau_{xxx}^{(1)}\frac{1}{5} z\left(3\frac{\partial^3 w_0}{\partial x \partial y^2} - 2\frac{\partial^3 w_0}{\partial x^3}\right) \\ & +\tau_{yyy}^{(1)}\frac{1}{5} z\left(3\frac{\partial^3 w_0}{\partial x^2 \partial y} - 2\frac{\partial^3 w_0}{\partial y^3}\right) \\ & +\tau_{zzz}^{(1)}\frac{1}{5}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right) \\ & +3\tau_{xxy}^{(1)}\frac{1}{5} z\left(\frac{\partial^3 w_0}{\partial y^3} - 4\frac{\partial^3 w_0}{\partial x^2 \partial y}\right) \\ & +3\tau_{xxz}^{(1)}\frac{1}{15}\left(\frac{\partial^2 w_0}{\partial y^2} - 4\frac{\partial^2 w_0}{\partial x^2}\right) \\ & +3\tau_{yyx}^{(1)}\frac{1}{5} z\left(\frac{\partial^3 w_0}{\partial x^3} - 4\frac{\partial^3 w_0}{\partial x \partial y^2}\right) \\ & +3\tau_{yyz}^{(1)}\frac{1}{15}\left(\frac{\partial^2 w_0}{\partial x^2} - 4\frac{\partial^2 w_0}{\partial y^2}\right) \\ & +3\tau_{zzx}^{(1)}\frac{1}{5} z\left(\frac{\partial^3 w_0}{\partial x^3} + \frac{\partial^3 w_0}{\partial x \partial y^2}\right) \\ & +3\tau_{zzy}^{(1)}\frac{1}{5} z\left(\frac{\partial^3 w_0}{\partial y^3} + \frac{\partial^3 w_0}{\partial x^2 \partial y}\right) - 2\tau_{xyz}^{(1)}\frac{\partial^2 w_0}{\partial x \partial y} \\ & +m_{xx}^s\frac{\partial^2 w_0}{\partial x \partial y} - m_{yy}^s\frac{\partial^2 w_0}{\partial x \partial y} \\ & +m_{xy}^s\left(\frac{\partial^2 w_0}{\partial y^2} - \frac{\partial^2 w_0}{\partial x^2}\right)\Bigg] dxdydz \end{aligned} \tag{21}$$

The kinetic energy is written as follows

$$K = \int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^b\int_0^a \frac{1}{2}\rho\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] dxdydz \tag{22}$$

The stress resultants are defined as follows

$$\begin{gathered} M_{xx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} z dz, \qquad M_{yy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yy} z dz, \\ M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xy} z dz, \qquad P_x^z = \int_{-\frac{h}{2}}^{\frac{h}{2}} p_x z dz, \end{gathered} \tag{23}$$

$$
\begin{aligned}
&P_y^z=\int_{-\frac{h}{2}}^{\frac{h}{2}} p_y z dz, \qquad P_z=\int_{-\frac{h}{2}}^{\frac{h}{2}} p_z dz,\\
&M_{xxx}^z=\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xxx}^{(1)} z dz, \quad M_{yyy}^z=\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yyy}^{(1)} z dz,\\
&M_{zzz}=\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zzz}^{(1)} dz, \quad M_{xxy}^z=\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xxy}^{(1)} z dz,\\
&M_{xxz}=\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xxz}^{(1)} dz, \quad M_{yyx}^z=\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yyx}^{(1)} z dz,\\
&M_{yyz}=\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yyz}^{(1)} dz, \quad M_{zzx}^z=\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zzx}^{(1)} z dz,\\
&M_{zzy}^z=\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zzy}^{(1)} z dz, \quad M_{xyz}=\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xyz}^{(1)} dz,\\
&Y_{xx}=\int_{-\frac{h}{2}}^{\frac{h}{2}} m_{xx} dz, \qquad Y_{yy}=\int_{-\frac{h}{2}}^{\frac{h}{2}} m_{yy} dz,\\
&Y_{xy}=\int_{-\frac{h}{2}}^{\frac{h}{2}} m_{xy} dz
\end{aligned}
\tag{23}
$$

The strain energy is rewritten as follows

$$
\begin{aligned}
U=&\frac{1}{2}\int_0^b\int_0^a\Bigg[-M_{xx}\frac{\partial^2 w_0}{\partial x^2}-M_{yy}\frac{\partial^2 w_0}{\partial y^2}-2M_{xy}\frac{\partial^2 w_0}{\partial x\partial y}\\
&-P_x^z\left(\frac{\partial^3 w_0}{\partial x^3}+\frac{\partial^3 w_0}{\partial x\partial y^2}\right)-P_y^z\left(\frac{\partial^3 w_0}{\partial y^3}+\frac{\partial^3 w_0}{\partial x^2\partial y}\right)\\
&-P_z\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial y^2}\right)+\frac{M_{xxx}^z}{5}\left(3\frac{\partial^3 w_0}{\partial x\partial y^2}-2\frac{\partial^3 w_0}{\partial x^3}\right)\\
&+\frac{M_{yyy}^z}{5}\left(3\frac{\partial^3 w_0}{\partial x^2\partial y}-2\frac{\partial^3 w_0}{\partial y^3}\right)+\frac{M_{zzz}}{5}\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial y^2}\right)\\
&+\frac{3M_{xxy}^z}{5}\left(\frac{\partial^3 w_0}{\partial y^3}-4\frac{\partial^3 w_0}{\partial x^2\partial y}\right)+\frac{M_{xxz}}{5}\left(\frac{\partial^2 w_0}{\partial y^2}-4\frac{\partial^2 w_0}{\partial x^2}\right)\\
&+\frac{3M_{yyx}^z}{5}\left(\frac{\partial^3 w_0}{\partial x^3}-4\frac{\partial^3 w_0}{\partial x\partial y^2}\right)+\frac{M_{yyz}}{5}\left(\frac{\partial^2 w_0}{\partial x^2}-4\frac{\partial^2 w_0}{\partial y^2}\right)\\
&+\frac{3M_{zzx}^z}{5}\left(\frac{\partial^3 w_0}{\partial x^3}+\frac{\partial^3 w_0}{\partial x\partial y^2}\right)+\frac{3M_{zzy}^z}{5}\left(\frac{\partial^3 w_0}{\partial y^3}+\frac{\partial^3 w_0}{\partial x^2\partial y}\right)\\
&-2M_{xyz}\frac{\partial^2 w_0}{\partial x\partial y}+Y_{xx}\frac{\partial^2 w_0}{\partial x\partial y}-Y_{yy}\frac{\partial^2 w_0}{\partial x\partial y}\\
&+Y_{xy}\left(\frac{\partial^2 w_0}{\partial y^2}-\frac{\partial^2 w_0}{\partial x^2}\right)\Bigg]dxdy
\end{aligned}
\tag{24}
$$

The inertia and stiffness terms are considered as follows

$$
\begin{aligned}
&\left\{A^{(1)},D^{(1)}\right\}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{g_1(z)}{1-\nu^2}\begin{Bmatrix}1\\ z^2\end{Bmatrix}dz\\
&\left\{A^{(2)},D^{(2)}\right\}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{g_1(z)\nu}{1-\nu^2}\begin{Bmatrix}1\\ z^2\end{Bmatrix}dz\\
&\left\{A^{(3)},D^{(3)}\right\}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{g_1(z)}{1+\nu}\begin{Bmatrix}1\\ z^2\end{Bmatrix}dz\\
&\begin{Bmatrix}I_1\\ I_3\end{Bmatrix}=\int_{-\frac{h}{2}}^{\frac{h}{2}}g_2(z)\begin{Bmatrix}1\\ z^2\end{Bmatrix}dz
\end{aligned}
\tag{25}
$$

Kinetic energy is also written as follows

$$
\begin{aligned}
K=&\int_0^b\int_0^a\frac{1}{2}f_2(x)h_2(y)\left[I_3\left(\frac{\partial^2 w_0}{\partial x\partial t}\right)^2\right.\\
&\left.+I_3\left(\frac{\partial^2 w_0}{\partial y\partial t}\right)^2+I_1\left(\frac{\partial w_0}{\partial t}\right)^2\right]dxdy
\end{aligned}
\tag{26}
$$

The governing equations and boundary conditions are obtained according to Hamilton's Principle

$$
\int_{t_1}^{t_2}(\delta U-\delta K)dt=0 \tag{27}
$$

The governing equation is obtained as follows

$$
\begin{aligned}
&-\frac{\partial^2 M_{xx}}{\partial x^2}-\frac{\partial^2 M_{yy}}{\partial y^2}-\frac{\partial^2 P_z}{\partial x^2}+\frac{\partial^3 P_x^z}{\partial x^3}\\
&+\frac{\partial^3 P_x}{\partial x\partial y^2}+\frac{\partial^3 P_y^z}{\partial y^3}+\frac{\partial^3 P_y^z}{\partial x^2\partial y}+\frac{\partial^2 Y_{xx}}{\partial x\partial y}\\
&-\frac{\partial^2 Y_{yy}}{\partial x\partial y}+\frac{\partial^2 Y_{xy}}{\partial y^2}-\frac{\partial^2 Y_{xy}}{\partial x^2}-\frac{\partial^2 P_z}{\partial y^2}\\
&+\frac{1}{5}\left[\frac{\partial^2 M_{zzz}}{\partial x^2}+\frac{\partial^2 M_{zzz}}{\partial y^2}+\frac{\partial^2 M_{xxz}}{\partial y^2}+\frac{\partial^2 M_{yyz}}{\partial x^2}\right]\\
&+\frac{2}{5}\left[\frac{\partial^3 M_{xxx}^z}{\partial x^3}+\frac{\partial^3 M_{yyy}^z}{\partial y^3}\right]\\
&-\frac{3}{5}\left[\frac{\partial^3 M_{xxx}^z}{\partial x\partial y^2}+\frac{\partial^3 M_{yyy}^z}{\partial x^2\partial y}+\frac{\partial^3 M_{xxy}^z}{\partial y^3}+\frac{\partial^3 M_{yyx}^z}{\partial x^3}\right.\\
&\left.+\frac{\partial^3 M_{zzx}^z}{\partial x^3}+\frac{\partial^3 M_{zzx}}{\partial x\partial y^2}+\frac{\partial^3 M_{zzy}^z}{\partial y^3}+\frac{\partial^3 M_{zzy}^z}{\partial x^2\partial y}\right]\\
&-\frac{4}{5}\left[\frac{\partial^2 M_{xxzz}}{\partial x^2}+\frac{\partial^2 M_{yyz}}{\partial y^2}\right]+\frac{12}{5}\left[\frac{\partial^3 M_{xxy}^z}{\partial x^2\partial y}+\frac{\partial^3 M_{yyx}^z}{\partial x\partial y^2}\right]\\
&-2\left[\frac{\partial^2 M_{xyz}}{\partial x\partial y}+\frac{\partial^2 M_{xy}}{\partial x\partial y}\right]\\
&=I_3h_2(y)\frac{\partial f_2(x)}{\partial x}\frac{\partial^3 w}{\partial x\partial t^2}+I_3f_2(x)\frac{\partial h_2(y)}{\partial y}\frac{\partial^3 w}{\partial y\partial t^2}\\
&+I_3f_2(x)h_2(y)\frac{\partial^4 w}{\partial x^2\partial t^2}-I_1f_2(x)h_2(y)\frac{\partial^2 w}{\partial t^2}\\
&+I_3f_2(x)h_2(y)\frac{\partial^4 w}{\partial y^2\partial t^2}
\end{aligned}
\tag{28}
$$

Also, the boundary conditions expressed as follow at $x=0,a$

$$
\begin{aligned}
&2\left[\frac{\partial M_{xy}}{\partial y}+\frac{\partial M_{xyz}}{\partial y}\right]+\frac{\partial M_{xx}}{\partial x}+\frac{\partial P_z}{\partial x}+\frac{\partial Y_{yy}}{\partial y}\\
&+\frac{\partial Y_{xy}}{\partial x}-\frac{\partial^2 P_x^z}{\partial x^2}-\frac{\partial^2 P_x^z}{\partial y^2}-\frac{\partial^2 P_x^z}{\partial x\partial y}-\frac{\partial Y_{xx}}{\partial y}\\
&+\frac{3}{5}\left[\frac{\partial^2 M_{xxx}^z}{\partial y^2}+\frac{\partial^2 M_{yyy}^z}{\partial x\partial y}+\frac{\partial^2 M_{yyx}^z}{\partial x^2}\right.\\
&\left.+\frac{\partial^2 M_{zzx}^z}{\partial x^2}+\frac{\partial^2 M_{zzx}^z}{\partial y^2}+\frac{\partial^2 M_{zzy}^z}{\partial x\partial y}\right]\\
&-\frac{1}{5}\left[\frac{\partial M_{zzz}}{\partial x}+\frac{\partial M_{yyz}}{\partial x}\right]-\frac{12}{5}\left[\frac{\partial^2 M_{xxy}^z}{\partial x\partial y}+\frac{\partial^2 M_{yyx}^z}{\partial y^2}\right]
\end{aligned}
\tag{29}
$$

$$-\frac{2}{5}\frac{\partial^2 M_{xxx}^z}{\partial x^2}+\frac{4}{5}\frac{\partial M_{xxz}}{\partial x}=0,\text{ or }\ \delta w_0=0 \tag{29}$$

$$\begin{aligned}&-M_{xx}+\frac{\partial P_x^z}{\partial x}+\frac{\partial P_y^z}{\partial y}-P_z+\frac{2}{5}\frac{\partial M_{xxx}^z}{\partial x}\\&-\frac{3}{5}\left[\frac{\partial M_{yyy}^z}{\partial y}+\frac{\partial M_{yyx}^z}{\partial x}+\frac{\partial M_{zzx}^z}{\partial x}+\frac{\partial M_{zzy}^z}{\partial y}\right]\\&+\frac{1}{5}\left[M_{zzz}+M_{yyz}\right]+\frac{12}{5}\frac{\partial M_{xxy}^z}{\partial y}\\&-\frac{4}{5}M_{xxz}-Y_{xy}=0,\quad\text{or}\quad \delta\frac{dw_0}{dx}=0\end{aligned} \tag{30}$$

$$\begin{aligned}&-P_x^z-\frac{1}{5}\left[2M_{xxx}^z-3M_{yyx}^z-3M_{zzx}^z\right]=0,\\&\text{or}\quad \delta\frac{d^2w_0}{dx^2}=0\end{aligned} \tag{31}$$

at $y=0,b$

$$\begin{aligned}&2\left[\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{xyz}}{\partial x}\right]+\frac{\partial M_{yy}}{\partial y}+\frac{\partial P_z}{\partial y}+\frac{\partial Y_{yy}}{\partial x}\\&+\frac{\partial Y_{xy}}{\partial y}-\frac{\partial^2 P_y^z}{\partial y^2}-\frac{\partial^2 P_y^z}{\partial x^2}-\frac{\partial^2 P_y^z}{\partial x\partial y}-\frac{\partial Y_{xx}}{\partial x}\\&+\frac{3}{5}\left[\frac{\partial^2 M_{yyy}^z}{\partial x^2}+\frac{\partial^2 M_{xxx}^z}{\partial x\partial y}+\frac{\partial^2 M_{xxy}^z}{\partial y^2}\right.\\&\left.+\frac{\partial^2 M_{zzy}^z}{\partial y^2}+\frac{\partial^2 M_{zzy}^z}{\partial x^2}+\frac{\partial^2 M_{zzx}^z}{\partial x\partial y}\right]\\&-\frac{1}{5}\left[+\frac{\partial M_{zzz}}{\partial y}+\frac{\partial M_{xxz}}{\partial y}+\right]-\frac{12}{5}\left[\frac{\partial^2 M_{yyx}^z}{\partial x\partial y}+\frac{\partial^2 M_{xxy}^z}{\partial x^2}\right]\\&-\frac{2}{5}\frac{\partial^2 M_{yyy}^z}{\partial y^2}+\frac{4}{5}\frac{\partial M_{yyz}}{\partial y}=0,\quad\text{or}\quad \delta w_0=0\end{aligned} \tag{32}$$

$$\begin{aligned}&-M_{yy}+\frac{\partial P_x^z}{\partial x}+\frac{\partial P_y^z}{\partial y}-P_z+\frac{2}{5}\frac{\partial M_{yyy}^z}{\partial y}\\&-\frac{3}{5}\left[\frac{\partial M_{xxx}^z}{\partial x}+\frac{\partial M_{xxy}^z}{\partial y}+\frac{\partial M_{zzx}^z}{\partial x}+\frac{\partial M_{zzy}^z}{\partial y}\right]\\&+\frac{1}{5}\left[M_{zzz}+M_{xxz}\right]+\frac{12}{5}\frac{\partial M_{yyx}^z}{\partial x}\\&-\frac{4}{5}M_{yyz}+Y_{xy}=0,\quad\text{or}\quad \delta\frac{dw_0}{dy}=0\end{aligned} \tag{33}$$

$$\begin{aligned}&-P_y^z-\frac{1}{5}\left[2M_{yyy}^z-3M_{xxy}^z-3M_{zzy}^z\right]=0,\\&\text{or}\quad \delta\frac{d^2w_0}{dy^2}=0\end{aligned} \tag{34}$$

For free vibration analysis, it is assumed that w_0 varies harmonically with respect to the time variable t, as follows

$$w_0(x,y,t)=\bar{w}(x,y)e^{i\omega t} \tag{35}$$

As a result, based on the MSGT, the governing equations and boundary conditions obtained for the Kirchhoff nanoplate made of a three-directional FGM. If the length scale parameters of the material l_0, l_1, l_2, are considered to be zero, then the governing equations and boundary conditions for the Kirchhoff nanoplate made of a three-directional FGM will be obtained based on classical theory (CT). Also, the governing equations for the Kirchhoff nanoplate are obtained based on the modified couple stress theory (MCST), which $l_0=l_1=0$ and $l_2=1$.

The following dimensionless quantities are defined

$$\begin{aligned}&\zeta=\frac{x}{a},\qquad \eta=\frac{y}{b},\qquad \xi=\frac{2z}{h},\\&\mu_1=\frac{E_m}{E_c},\qquad \mu_2=\frac{\rho_m}{\rho_c},\qquad W=\frac{w}{h},\\&(\ell_0,\ell_1,\ell_2)=\frac{(l_0,l_1,l_2)}{h},\qquad \varphi=\frac{a}{b},\qquad \gamma=\frac{h}{a}\end{aligned} \tag{36}$$

By substituting Eq. (35) in Eqs. (28) to (34) and using the Eq. (36) the dimensionless equilibrium equation and boundary conditions are obtained by the displacement term.

Boundary conditions at $\zeta=0,1$

$$\begin{aligned}&\left(\ell_0^2+\frac{2}{5}\ell_1^2\right)\nu_1\frac{\partial^5 W}{\partial\zeta^5}+\left(2\ell_0^2+\frac{4}{5}\ell_1^2\right)\beta_1\nu_1\frac{\partial^4 W}{\partial\zeta^4}\\&-\left[\frac{1}{\gamma^2}-\left(\ell_0^2+\frac{2}{5}\ell_1^2\right)\beta_1^2\nu_1-\left(\ell_0^2-\frac{3}{5}\ell_1^2\right)\beta_2^2\varphi^2\nu_1\right.\\&\left.-\left(12\ell_0^2+\frac{16}{5}\ell_1^2+6\ell_2^2\right)\frac{\nu_1 f_0}{\gamma^2 f_2}\right]\frac{\partial^3 W}{\partial\zeta^3}\\&-\left[-\frac{1}{\gamma^2}+\left(12\ell_0^2+\frac{16}{5}\ell_1^2+6\ell_2^2\right)\frac{\nu_1 f_0}{\gamma^2 f_2}\right]\beta_1\frac{\partial^2 W}{\partial\zeta^2}\\&+\left(3\ell_0^2+\frac{6}{5}\ell_1^2\right)\nu_1\varphi^2\frac{\partial^5 W}{\partial\zeta^3\partial\eta^2}+\left(2\ell_0^2+\frac{9}{5}\ell_1^2\right)\nu_1\varphi^4\frac{\partial^5 W}{\partial\zeta\partial\eta^4}\\&+\left(3\ell_0^2+\frac{6}{5}\ell_1^2\right)\beta_2\nu_1\varphi^2\frac{\partial^4 W}{\partial\zeta^3\partial\eta}\\&+\left(3\ell_0^2+\frac{21}{5}\ell_1^2\right)\beta_2\nu_1\varphi^4\frac{\partial^4 W}{\partial\zeta\partial\eta^3}\\&+\left(3\ell_0^2+\frac{6}{5}\ell_1^2\right)\beta_1\nu_1\varphi^2\frac{\partial^4 W}{\partial\zeta^2\partial\eta^2}\\&+\left(\ell_0^2+\frac{12}{5}\ell_1^2\right)\beta_1\beta_2\nu_1\varphi^2\frac{\partial^3 W}{\partial\zeta^2\partial\eta}\\&+\left[-\frac{(\nu+\nu_1)}{\gamma^2}+\left(\ell_0^2-\frac{3}{5}\ell_1^2\right)\beta_1^2\nu_1\right.\\&+\left(\ell_0^2+\frac{12}{5}\ell_1^2\right)\beta_2^2\nu_1\varphi^2\\&\left.-\left(12\ell_0^2+\frac{36}{5}\ell_1^2+18\ell_2^2\right)\frac{\nu_1 f_0}{\gamma^2 f_2}\right]\varphi^2\frac{\partial^3 W}{\partial\zeta\partial\eta^2}\\&-2\left[\frac{\varphi^2}{\gamma^2}+\left(4\ell_1^2+12\ell_2^2\right)\frac{\nu_1\varphi^2 f_0}{\gamma^2 f_2}\right]\beta_2\frac{\partial^2 W}{\partial\zeta\partial\eta}\\&-\left[\frac{\nu}{\gamma^2}+\left(12\ell_0^2-\frac{4}{5}\ell_1^2-6\ell_2^2\right)\frac{\nu_1 f_0}{\gamma^2 f_2}\right]\beta_1\varphi^2\frac{\partial^2 W}{\partial\eta^2}\\&+\left(\ell_0^2-\frac{3}{5}\ell_1^2\right)\beta_1\beta_2\nu_1\varphi^4\frac{\partial^3 W}{\partial\eta^3}\\&+\left(\ell_0^2-\frac{3}{5}\ell_1^2\right)\beta_1\nu_1\varphi^4\frac{\partial^4 W}{\partial\eta^4}=0,\quad\text{or}\quad \delta W{=}0\end{aligned} \tag{37}$$

$$\begin{aligned}&-\left(\ell_0^2+\frac{2}{5}\ell_1^2\right)\nu_1\gamma^2\frac{\partial^4 W}{\partial\zeta^4}-\left(\ell_0^2+\frac{2}{5}\ell_1^2\right)\beta_1\nu_1\gamma^2\frac{\partial^3 W}{\partial\zeta^3}\\&+\left[1+\left(12\ell_0^2+\frac{16}{5}\ell_1^2+6\ell_2^2\right)\frac{\nu_1 f_0}{f_2}\right]\frac{\partial^2 W}{\partial\zeta^2}\\&-\left(2\ell_0^2+\frac{9}{5}\ell_1^2\right)\nu_1\gamma^2\varphi^2\frac{\partial^4 W}{\partial\zeta^2\partial\eta^2}\\&-\left(\ell_0^2+\frac{12}{5}\ell_1^2\right)\beta_2\nu_1\gamma^2\varphi^2\frac{\partial^3 W}{\partial\zeta^2\partial\eta}\end{aligned} \tag{38}$$

$$-\left(\ell_0^2+\frac{3}{5}\ell_1^2\right)\beta_1\nu_1\gamma^2\varphi^2\frac{\partial^3 W}{\partial\zeta\partial\eta^2}$$
$$+\left[\nu+\left(12\ell_0^2-\frac{4}{5}\ell_1^2-6\ell_2^2\right)\frac{\nu_1 f_0}{f_2}\right]\varphi^2\frac{\partial^2 W}{\partial\eta^2}$$
$$-\left(\ell_0^2-\frac{3}{5}\ell_1^2\right)\beta_2\nu_1\gamma^2\varphi^4\frac{\partial^3 W}{\partial\eta^3}$$
$$-\left(\ell_0^2-\frac{3}{5}\ell_1^2\right)\nu_1\gamma^2\varphi^4\frac{\partial^4 W}{\partial\eta^4}=0,\quad \text{or}\quad \delta\frac{dW}{d\zeta}=0 \tag{38}$$

$$M_{xx}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{xx}z\,dz \tag{39}$$

at $\eta=0,1$

$$\left(\ell_0^2+\frac{2}{5}\ell_1^2\right)\nu_1\varphi^5\frac{\partial^5 W}{\partial\eta^5}+\left(2\ell_0^2+\frac{4}{5}\ell_1^2\right)\beta_2\nu_1\varphi^5\frac{\partial^4 W}{\partial\eta^4}$$
$$-\left[\frac{1}{\gamma^2}-\left(\ell_0^2-\frac{3}{5}\ell_1^2\right)\beta_1^2\nu_1-\left(\ell_0^2+\frac{2}{5}\ell_1^2\right)\beta_2^2\varphi^2\nu_1\right.$$
$$\left.-\left(12\ell_0^2+\frac{16}{5}\ell_1^2+6\ell_2^2\right)\frac{\nu_1 f_0}{\gamma^2 f_2}\right]\varphi^3\frac{\partial^3 W}{\partial\eta^3}$$
$$-\left[-\frac{1}{\gamma^2}+\left(12\ell_0^2+\frac{16}{5}\ell_1^2+6\ell_2^2\right)\frac{\nu_1 f_0}{\gamma^2 f_2}\right]\beta_2\varphi^3\frac{\partial^2 W}{\partial\eta^2}$$
$$+\left(\ell_0^2-\frac{3}{5}\ell_1^2\right)\beta_2\nu_1\varphi\frac{\partial^4 W}{\partial\zeta^4}+\left(\ell_0^2-\frac{3}{5}\ell_1^2\right)\beta_1\beta_2\nu_1\varphi\frac{\partial^3 W}{\partial\zeta^3}$$
$$-\left[1+\left(12\ell_0^2-\frac{4}{5}\ell_1^2-6\ell_2^2\right)\frac{\nu_1 f_0}{\gamma^2 f_2}\right]\frac{\beta_2\varphi}{\gamma^2}\frac{\partial^2 W}{\partial\zeta^2}$$
$$+\left(2\ell_0^2+\frac{9}{5}\ell_1^2\right)\nu_1\varphi\frac{\partial^5 W}{\partial\zeta^4\partial\eta}+\left(3\ell_0^2+\frac{6}{5}\ell_1^2\right)\nu_1\varphi^3\frac{\partial^5 W}{\partial\zeta^2\partial\eta^3}$$
$$+\left(3\ell_0^2+\frac{21}{5}\ell_1^2\right)\beta_1\nu_1\varphi\frac{\partial^4 W}{\partial\zeta^3\partial\eta}$$
$$+\left(3\ell_0^2+\frac{6}{5}\ell_1^2\right)\beta_2\nu_1\varphi^3\frac{\partial^4 W}{\partial\zeta^2\partial\eta^2}$$
$$+\left(3\ell_0^2+\frac{6}{5}\ell_1^2\right)\beta_1\nu_1\varphi^3\frac{\partial^4 W}{\partial\zeta\partial\eta^3}$$
$$+\left[-\frac{(\nu+\nu_1)}{\gamma^2}+\left(\ell_0^2+\frac{12}{5}\ell_1^2\right)\beta_1^2\nu_1+\left(\ell_0^2-\frac{3}{5}\ell_1^2\right)\beta_2^2\varphi^2\nu_1\right.$$
$$\left.-\left(12\ell_0^2+\frac{36}{15}\ell_1^2+18\ell_2^2\right)\frac{\nu_1 f_0}{\gamma^2 f_2}\right]\varphi\frac{\partial^3 W}{\partial\zeta^2\partial\eta}$$
$$+\left(\ell_0^2+\frac{12}{5}\ell_1^2\right)\beta_1\beta_2\varphi^3\frac{\partial^3 W}{\partial\zeta\partial\eta^2}$$
$$-2\left[\frac{\nu_1}{\gamma^2}+\left(4\ell_1^2+12\ell_2^2\right)\frac{\nu_1 f_0}{\gamma^2 f_2}\right]\beta_1\varphi\frac{\partial^2 W}{\partial\zeta\partial\eta}=0,\quad \text{or}\quad \delta\mathrm{W}=0 \tag{40}$$

$$-\left(\ell_0^2-\frac{3}{5}\ell_1^2\right)\nu_1\gamma^2\frac{\partial^4 W}{\partial\zeta^4}-\left(\ell_0^2-\frac{3}{5}\ell_1^2\right)\beta_1\nu_1\gamma^2\frac{\partial^3 W}{\partial\zeta^3}$$
$$+\left[\nu+\left(12\ell_0^2-\frac{4}{5}\ell_1^2-6\ell_2^2\right)\frac{\nu_1 f_0}{f_2}\right]\frac{\partial^2 W}{\partial\zeta^2}$$
$$-\left(2\ell_0^2+\frac{9}{5}\ell_1^2\right)\nu_1\gamma^2\varphi^2\frac{\partial^4 W}{\partial\zeta^2\partial\eta^2}$$
$$-\left(\ell_0^2-\frac{3}{5}\ell_1^2\right)\beta_2\nu_1\gamma^2\varphi^2\frac{\partial^3 W}{\partial\zeta^2\partial\eta}$$
$$-\left(\ell_0^2+\frac{12}{5}\ell_1^2\right)\beta_1\nu_1\gamma^2\varphi^2\frac{\partial^3 W}{\partial\zeta\partial\eta^2}$$
$$+\left[1+\left(12\ell_0^2+\frac{16}{5}\ell_1^2+6\ell_2^2\right)\frac{\nu_1 f_0}{f_2}\right]\varphi^2\frac{\partial^2 W}{\partial\eta^2}$$
$$-\left(\ell_0^2+\frac{2}{5}\ell_1^2\right)\beta_2\nu_1\gamma^2\varphi^4\frac{\partial^3 W}{\partial\eta^3}$$
$$-\left(\ell_0^2+\frac{2}{5}\ell_1^2\right)\nu_1\gamma^2\varphi^4\frac{\partial^4 W}{\partial\eta^4}=0,\quad \text{or}\quad \delta\frac{dW}{d\eta}=0 \tag{41}$$

$$\left(\ell_0^2+\frac{2}{5}\ell_1^2\right)\varphi^2\frac{\partial^3 W}{\partial\eta^3}+\left(\ell_0^2-\frac{3}{5}\ell_1^2\right)\frac{\partial^3 W}{\partial\zeta^2\partial\eta}=0,$$
$$\text{or}\quad \delta\frac{d^2 W}{d\eta^2}=0 \tag{42}$$

where

$$f_0=\frac{1}{2}\int_{-1}^{0}[\mu_1+(1-\mu_1)(-\xi)^{n_1}]d\xi+\frac{1}{2}\int_{0}^{1}[\mu_1+(1-\mu_1)(\xi)^{n_1}]d\xi \tag{43}$$

$$f_2=\frac{3}{2}\int_{-1}^{0}\xi^2[\mu_1+(1-\mu_1)(-\xi)^{n_1}]d\xi+\frac{3}{2}\int_{0}^{1}\xi^2[\mu_1+(1-\mu_1)(\xi)^{n_1}]d\xi \tag{44}$$

$$g_0=\frac{1}{2}\int_{-1}^{0}[\mu_2+(1-\mu_2)(-\xi)^{n_2}]d\xi+\frac{1}{2}\int_{0}^{1}[\mu_2+(1-\mu_2)(\xi)^{n_2}]d\xi \tag{45}$$

$$g_2=\frac{3}{2}\int_{-1}^{0}\xi^2[\mu_2+(1-\mu_2)(-\xi)^{n_2}]d\xi+\frac{3}{2}\int_{0}^{1}\xi^2[\mu_2+(1-\mu_2)(\xi)^{n_2}]d\xi \tag{46}$$

$$\nu_1=1-\nu \tag{47}$$

$$\Omega^2=\frac{\rho_c a^2\omega^2}{E_c} \tag{48}$$

That Ω^2 is the dimensionless frequency of the Kirchhoff nanoplate made of a three-directional FGM.

3. Generalized Differential Quadrature Method (GDQM)

In the case of general boundary conditions, it is very difficult to analyze the Eq. (37). Therefore, the generalized GDQM is used to solve the Eq. (37). The GDQ method is an easy and useful tool for analyzing problems. In addition, the GDQM is an efficient numerical method for solving differential equations. In the GDQ method, the governing equations and boundary conditions are converted to discrete expressions. When the problem is discretized, the meshed points along *x* and *y* contain *N* and *M* nodes respectively, which are obtained from the Chebyshev-Gauss-Lubatto function

$$\zeta_i=\frac{1}{2}\left[1-cos\left(\frac{i-1}{N-1}.\pi\right)\right],\ i=1{:}N$$
$$\eta_j=\frac{W}{2}\left[1-cos\left(\frac{j-1}{M-1}.\pi\right)\right],\ j=1{:}M \tag{49}$$

In this method, the derivatives of function $f(\zeta,\eta)$ are obtained at the points of (ζ_i,η_j) in the form below

$$\begin{aligned}
&f_\zeta^{(n)}(\zeta_i,\eta_j)=\sum_{k=1}^{N}A_{ik}^{(n)}f(\zeta_k,\eta_j)\,,\ n=2,\dots,N-1\\
&f_\eta^{(m)}(\zeta_i,\eta_j)=\sum_{l=1}^{M}B_{jl}^{(m)}f(\zeta_i,\eta_l)\,,\ m=2,\dots,M-1\\
&f_{\zeta\eta}^{(n+m)}(\zeta_i,\eta_j)=\sum_{k=1}^{N}\sum_{l=1}^{M}A_{ik}^{(n)}B_{jl}^{(m)}f(\zeta_k,\eta_l)\,,\begin{cases}i=1,\dots,N\\ j=1,\dots,M\end{cases}
\end{aligned}\tag{50}$$

$A_{ik}^{(n)}$ and $B_{jl}^{(m)}$ are weighted coefficients for derivatives of order n and m in the direction of ζ and η respectively. The first order derivative of weighted coefficients are obtained as follows

$$\begin{aligned}
&A_{ij}^{(1)}=\frac{M(\zeta_i)}{(\zeta_i-\zeta_j)M(\zeta_j)}, && i\neq j\\
&A_{ii}^{(1)}=-\sum_{j=1,j\neq i}^{N}A_{ij}^{(1)}\,, && i,j=1,\dots,N\\
&B_{ij}^{(1)}=\frac{P(\eta_i)}{(\eta_i-\eta_j)P(\eta_j)}, && i\neq j\\
&B_{ii}^{(1)}=-\sum_{j=1,j\neq i}^{M}B_{ij}^{(1)}\,, && i,j=1,\dots,M\\
&M(\zeta_i)=\prod_{j=1,j\neq i}^{N}(\zeta_i-\zeta_j)\\
&P(\eta_i)=\prod_{j=1,j\neq i}^{M}(\eta_i-\eta_j)
\end{aligned}\tag{51}$$

The weighted coefficients for second order and higher derivatives are obtained as follows

$$\begin{aligned}
&A_{ij}^{(n)}=n\left[A_{ij}^{(1)}A_{ii}^{(n-1)}-\frac{A_{ij}^{(n-1)}}{\zeta_i-\zeta_j}\right], && i\neq j\\
&A_{ii}^{(n)}=-\sum_{j=1,j\neq i}^{N}A_{ij}^{(n)}\,, && i=j,\ n=2,\dots,N-1,\\
& && i,j=1,\dots,N\\
&B_{ij}^{(m)}=m\left[B_{ij}^{(1)}B_{ii}^{(m-1)}-\frac{B_{ij}^{(m-1)}}{\eta_i-\eta_j}\right], && i\neq j\\
&B_{ii}^{(m)}=-\sum_{j=1,j\neq i}^{M}B_{ij}^{(m)}\,, && i=j,\ \ m=2,\dots,M-1,\\
& && i,j=1,\dots,M
\end{aligned}\tag{52}$$

The Eq. (50) is inserted in the Eqs. (37)-(42). Then, by arrangement of the displacement variables and the corresponding coefficients, the governing equations are obtained as follows

$$\begin{bmatrix}K_{bb} & K_{bd}\\ K_{db} & K_{dd}\end{bmatrix}\begin{bmatrix}W_b\\ W_d\end{bmatrix}=\Omega^2\begin{bmatrix}0 & 0\\ M_{db} & M_{dd}\end{bmatrix}\begin{bmatrix}W_b\\ W_d\end{bmatrix}\tag{53}$$

The indexes of b and d represent the points inside the domain and boundary points, respectively. By removing boundary points, dimensions of the coefficients matrix are reduced by using boundary conditions. As a result, Eq. (53) is written as follows

$$[K][W_d]=\Omega^2[M][W_d]\tag{54}$$

where

$$[K]=[K_{dd}]-[K_{db}][K_{bb}]^{-1}[K_{bd}]\tag{55}$$

$$[M]=[M_{dd}]-[M_{db}][M_{bb}]^{-1}[M_{bd}]\tag{56}$$

By solving the mentioned eigenvalues problem, the dimensionless free vibration of the Kirchhoff nanoplate made of a three-directional FGM is obtained based on the MSGT.

4. Results and discussion

In the following, numerical results are presented for the analysis of vibrations of nanoplate made of three-directional FGMs with different boundary conditions. Results are presented based on CT, MCST and MSGT to compare predicted response by different theories. In this paper, five different boundary conditions for FG nanoplates are considered, which are as follows:

(1) CCCC: All edges are fixed supported
(2) SSSS: All edges are simply supported
(3) CCCS: Two edges in x direction are fixed supported and two edges in y direction are fixed supported for y = 0 and simply supported at y = b
(4) SSSC: Two edges in x direction are simply supported and two edges in y direction are simply supported for y = 0 and fixed supported at y = b
(5) SCSC: Two edges in x direction are simply supported and two edges in y direction are fixed supported

To validate the study, free vibration of homogeneous microplate is considered and compared with the study of Wang *et al*. (2011). To evaluate the study, all the scale parameters of material are considered equal, it means that $l_0=l_1=l_2=cl$.

In practice, the length scale parameter of the material cl is obtained from mechanical tests (tensile or axial pressure testing, bending test or torsion test) for different size samples. In the paper of Wang *et al*. (2011) the length scale parameter of the material is considered as $cl=0.5\,\mu\text{m}$. Also, the ratio of the dimensions of the microplate is considered equal (*a*/*h* = 50, *b*/*h* = 50) and the properties of the materials are considered as $E=1.44\,\text{GPa}$, $\rho=2000\,\text{kg/m}^3$ and $\nu=0.3$. To better describe the size effects, the dimensionless size parameter of k is defined, which is defined as the ratio of plate thickness to the length scale parameter of the material ($k=h/cl$).

In Fig. 2, the frequency of the first mode of homogeneous nanoplate is plotted for three theories of modified strain gradient, modified couple stress and classical theory. The first frequency of the present paper is

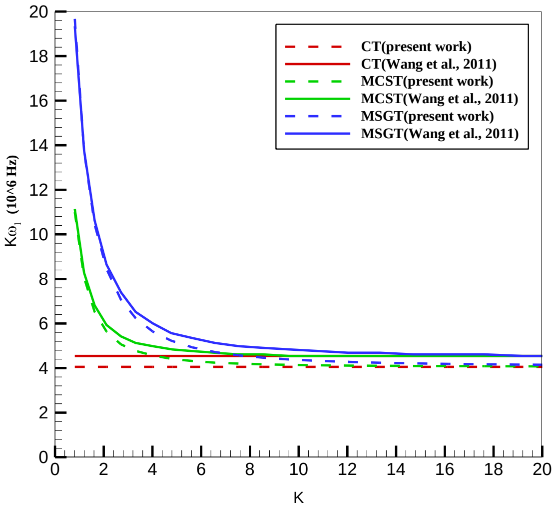


Fig. 2 Validation of normalized free vibrations of nano plate (SSSS) based on the size scale (Wang *et al*. 2011)

Table 1 Validation results of this paper with Thai & Choi paper's for homogeneous (SSSS) square plate

Mode	a/h	Ω(Mhz) (Thai and Choi 2013)	Ω(Mhz) (Present work)
1	5	1.6289	1.6297
	10	0.4170	0.4170
	20	0.1049	0.1049
	30	0.0467	0.0467
2	5	3.8958	3.8963
	10	1.0300	1.0300
	20	0.2614	0.2614
	30	0.1165	0.1165
3	5	5.9848	5.9852
	10	1.6289	1.6289
	20	0.4170	0.4170
	30	0.1862	0.1862

Table 2 Material properties used in the numerical study

Materials	Properties		
	E (MPa)	ν	ρ (kg/m^3)
Aluminum (AL)	70	0.3	2702
Zirconia (ZrO_2)	200	0.3	5700

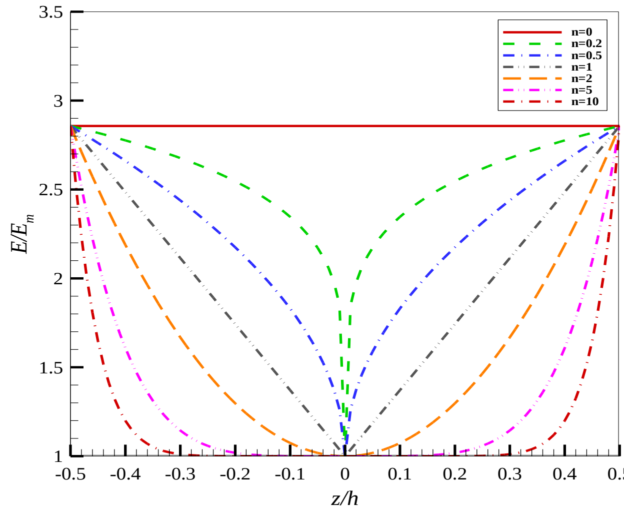
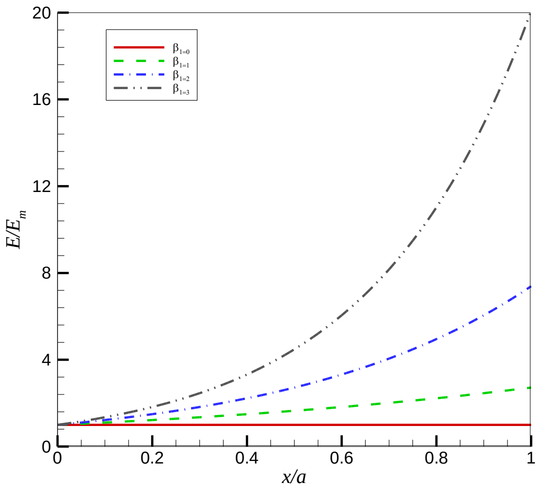


Fig. 3 Distribution of the dimensionless modulus of elasticity based on the dimensionless z/h parameter at $x, y = 0$

compared with Wang *et al*. (2011) results, in Fig. 2, which indicates that close accuracy has been achieved. We also observe that by increasing the dimensionless parameter k, the natural frequency decreases.

This paper is also compared with Thai and Choi paper (Thai and Choi 2013), which is a MCST, for homogeneous square nanoplate for the different dimensional ratios in Table 1. In this table can see that the results of this paper are too accurate.

In order to investigate the parameters of the problem in all of the results ahead, the metal and ceramic parts of



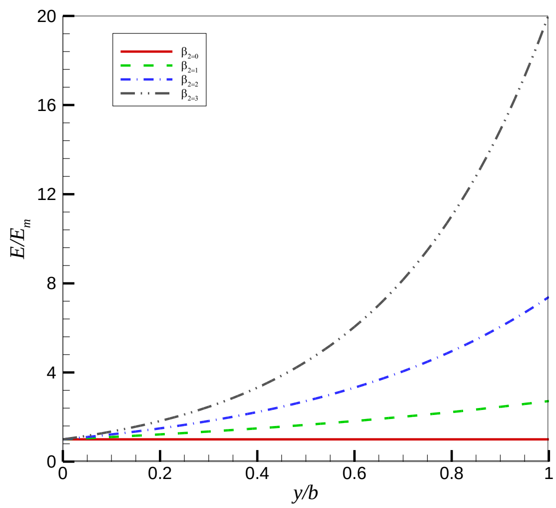


Fig. 4 Distribution of the dimensionless modulus of elasticity based on the dimensionless x/a and y/b at $z = -h/2$

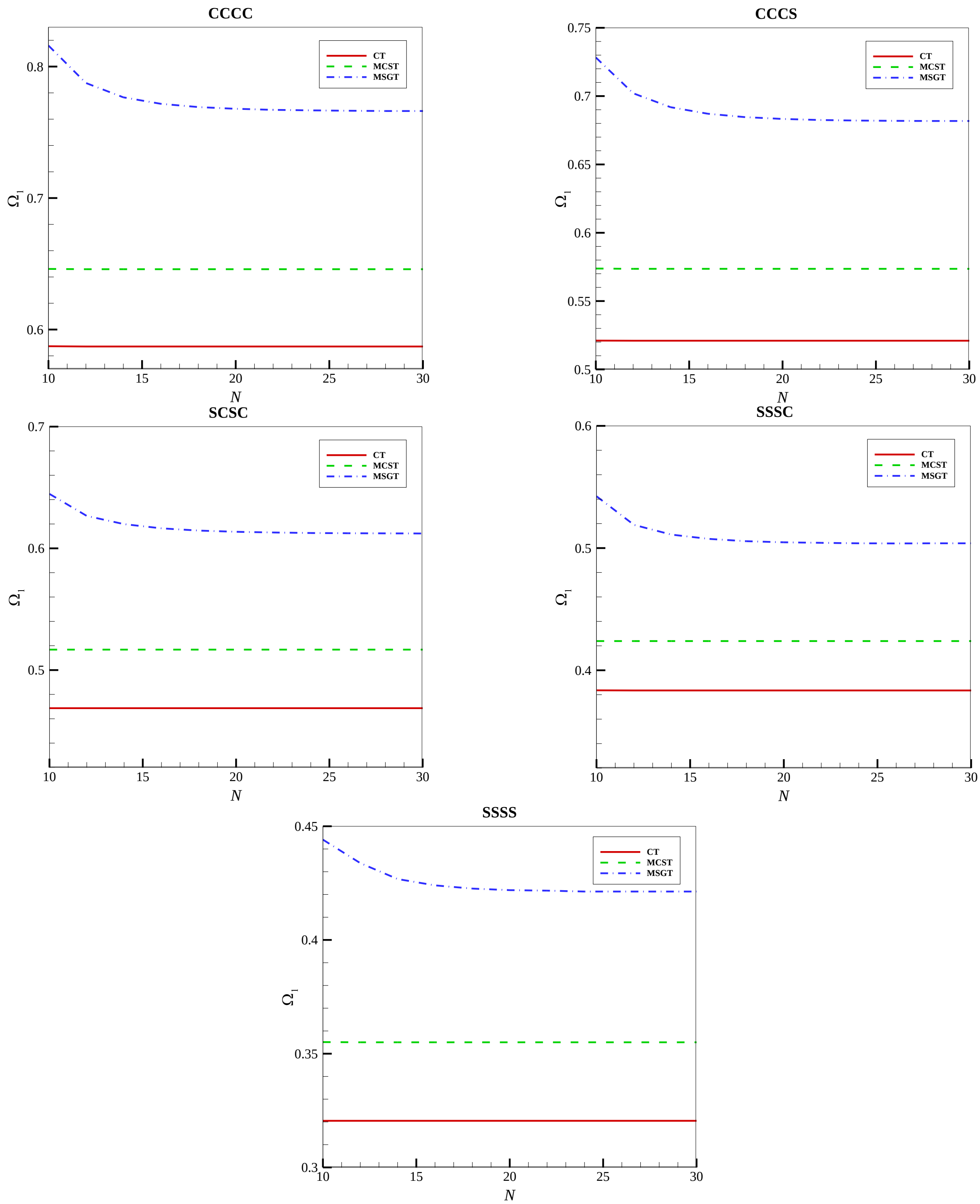


Fig. 5 Convergence of the dimensionless frequency in the first mode for various boundary conditions and three theories CT, MCST and MSGT

the FG material are considered as aluminum and zirconia, respectively. The properties of these materials are presented in Table 2 (Hosseini-Hashemi *et al*. 2011).

The elasticity modulus and density of nanoplate change in three directions of x, y and z, are considered as follows

$$E(x,z) = \begin{cases} e^{\frac{\beta_1}{a}x} e^{\frac{\beta_2}{b}y} \left[E_m + (E_c - E_m)\left(\frac{-2z}{h}\right)^{n_1} \right] \\ \text{for } \left(-\frac{h}{2} \le z \le 0\right) \\ e^{\frac{\beta_1}{a}x} e^{\frac{\beta_2}{b}y} \left[E_m + (E_c - E_m)\left(\frac{2z}{h}\right)^{n_1} \right] \\ \text{for } \left(0 \le z \le \frac{h}{2}\right) \end{cases} \quad (57)$$

$$\rho(x,z)=\begin{cases} e^{\frac{\beta_1}{a}x}e^{\frac{\beta_2}{b}y}\left[\rho_m+(\rho_c-\rho_m)\left(\frac{-2z}{h}\right)^{n_2}\right] \\ \text{for}\quad \left(-\frac{h}{2}\le z\le 0\right) \\ e^{\frac{\beta_1}{a}x}e^{\frac{\beta_2}{b}y}\left[\rho_m+(\rho_c-\rho_m)\left(\frac{2z}{h}\right)^{n_2}\right] \\ \text{for}\quad \left(0\le z\le \frac{h}{2}\right) \end{cases} \tag{57}$$

Figs. 3 and 4 show distribution of the elasticity modulus along the length, width and thickness of the nanoplates for different values of $n_1=n_2=n$ and β. If n is considered to be zero, the whole plate becomes ceramic and if n is infinite, it is completely metallic. Also, for $n = 1$, the ceramic and metal variations are linear. In Fig. 3, the distribution of the modulus of elasticity is plotted based on the z/h dimensionless parameter at $x,y=0$, which indicates the distribution of the modulus of elasticity along

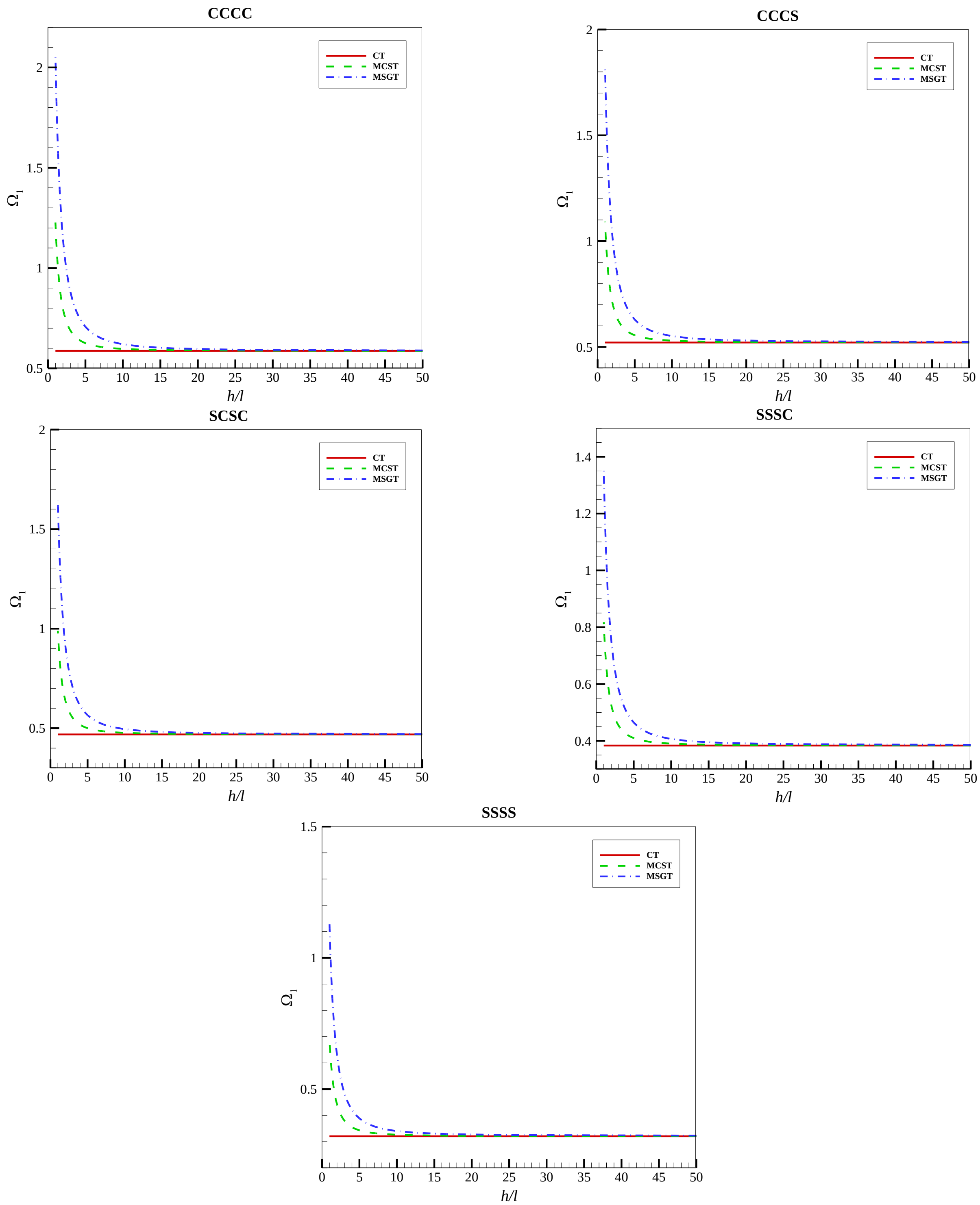


Fig. 7 The first dimensionless natural frequency of the FG squared nanoplate based on the material gradient index (n) on the basis of the MSGT, MCST and CT for $\beta_1=\beta_2=0$, $h/l=4$ and $h/a=0.05$

the thickness of the plate. In this figure, with the increase of the gradient index (n), the dimensionless modulus of elasticity decreases.

Also, in Fig. 4, the distribution of the dimensionless elastic modulus is plotted based on the dimensionless x/a and y/b parameter in $z = 0$, which indicates the distribution of the modulus of elasticity along the length and width of the plate. In this figure with increasing β, the elasticity modulus increases. Also, for $\beta = 0$, the distribution of the elasticity modulus is constant along $x$and y direction.

Fig. 5 indicates the convergence of the GDQM in obtaining the natural frequency in the first mode for various boundary conditions. It can be seen that considering more than 25 sample points does not significantly affect the accuracy of the results.

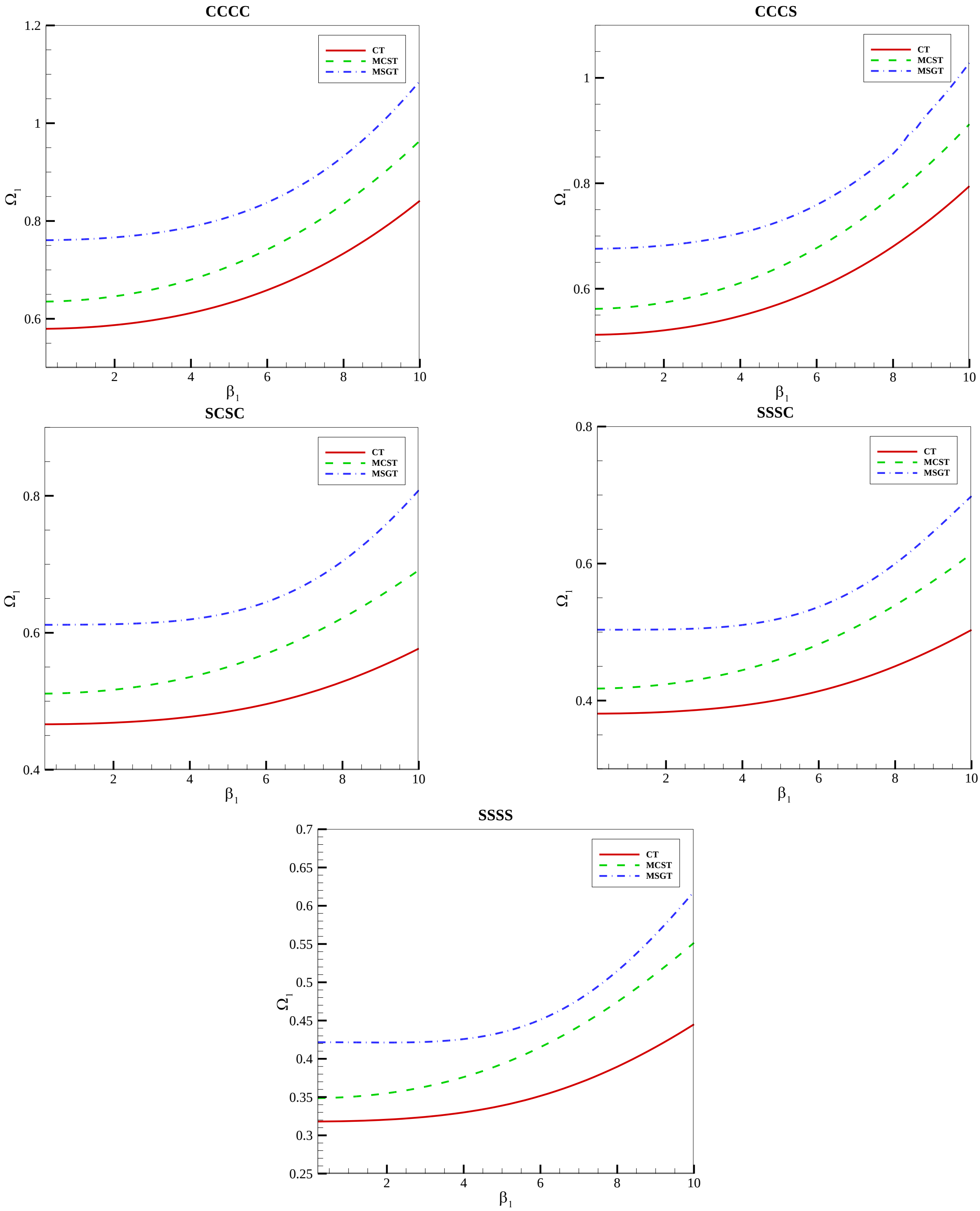


Fig. 8 The first dimensionless natural frequency of the FG squared nanoplate based on β_1 on the basis of the MSGT, MCST and CT for $n = 2$, $h/l = 4$, $\beta_2 = 0$ and $h/a = 0.05$

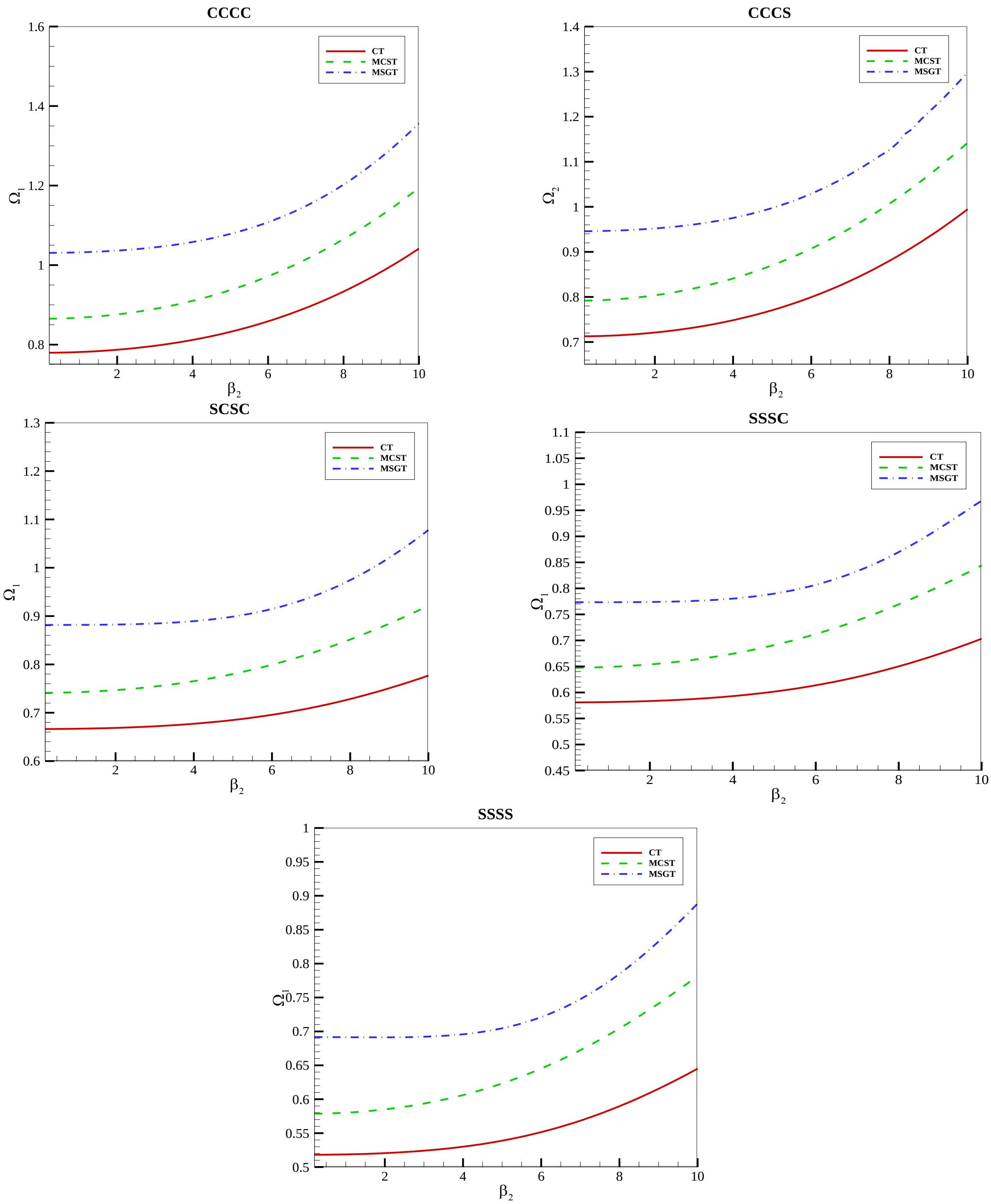


Fig. 9 The first dimensionless natural frequency of the FG squared nanoplate based on β_1 on the basis of the MSGT, MCST and CT for $n = 2$, $h/l = 4$, $\beta_1 = 10$ and $h/a = 0.05$

4.1 Effects of size scale

Fig. 6 shows the variations of dimensionless frequency of the FG nanoplate based on the dimensionless length scale parameter h/l for three theories of classical, modified couple stress and modified strain gradient for different boundary conditions. In the MSGT, if the length scale parameters of l_0 and l_1 consider to be zero and $l_2 = l$, it becomes the MCST, and if the l_0 and l_1 and l_2 are considered to be zero, it becomes classical theory. It should be noted that the dimensionless vibration response of the nanoplate on the basis of the classical theory remains unchanged due to the absence of the parameter of the length scale, while the dimensionless frequency is reduced by increasing the length scale parameter, based on the MCST and MSGT, and approach to the theory of classical stress.

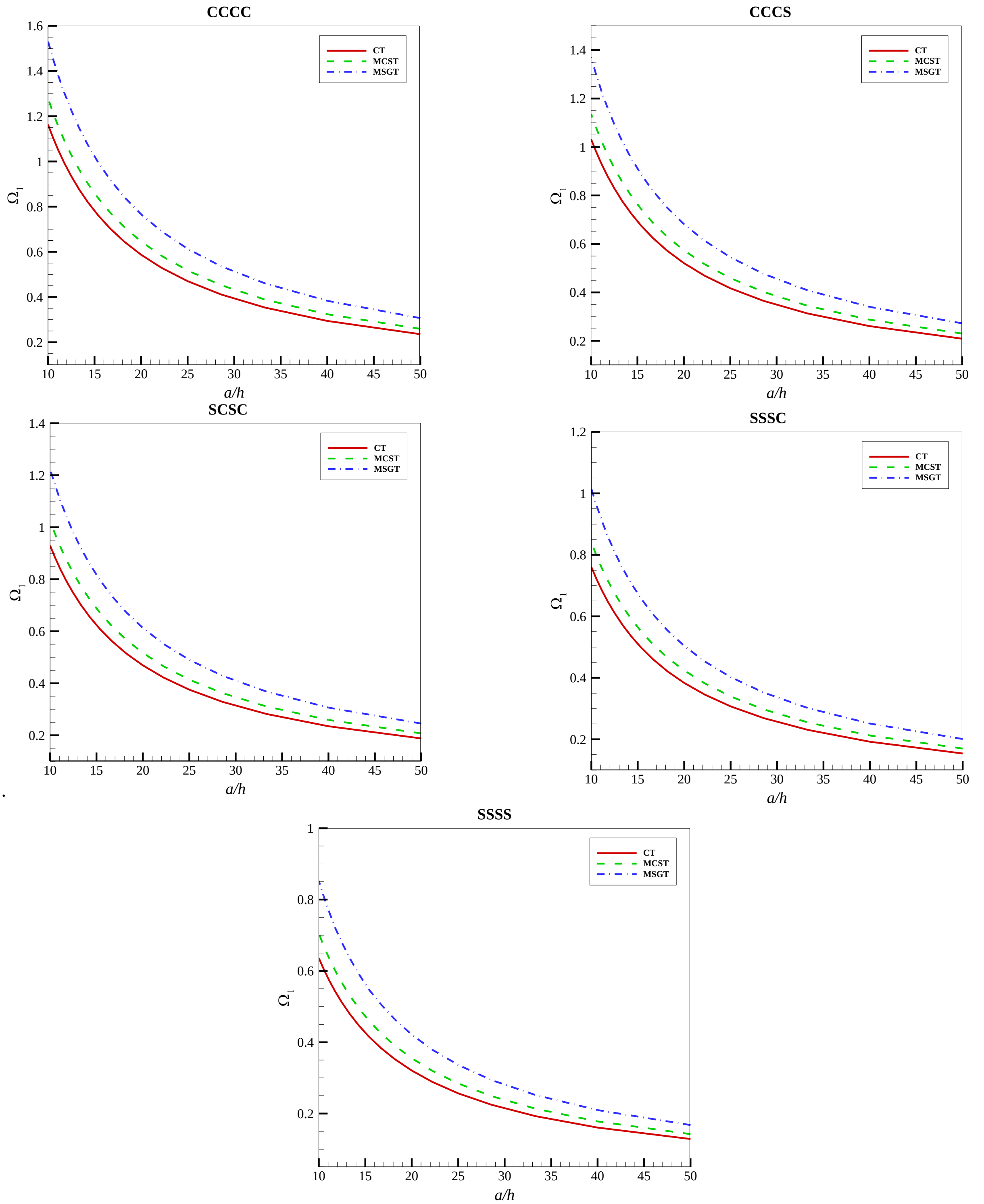


Fig. 10 The first dimensionless natural frequency of the FG squared nanoplate based on the aspect ratio of plate on the basis of the MSGT, MCST and CT for $n = 2$, $h/l = 4$ and $\beta_1 = \beta_2 = 0$

This means that by increasing the structural size, the effects of the size scale are eliminated. MCST and MSGT also predict the dimensionless frequency more than classical theory, and this pattern is more evident for a smaller length scale. It is also observed that the natural frequency for the 4-sided fixed supporting condition is maximum, and for the 4-sided simple supporting, has the minimum value, which is due to the fact that the fixed support is harder and the simple support is softer.

4.2 Effect of material gradient index

Fig. 7 shows the dimensionless frequency variations of the FG nanoplate based on the material gradient index (n) for three theories of classical, modified couple stress and modified strain gradient for different boundary conditions.

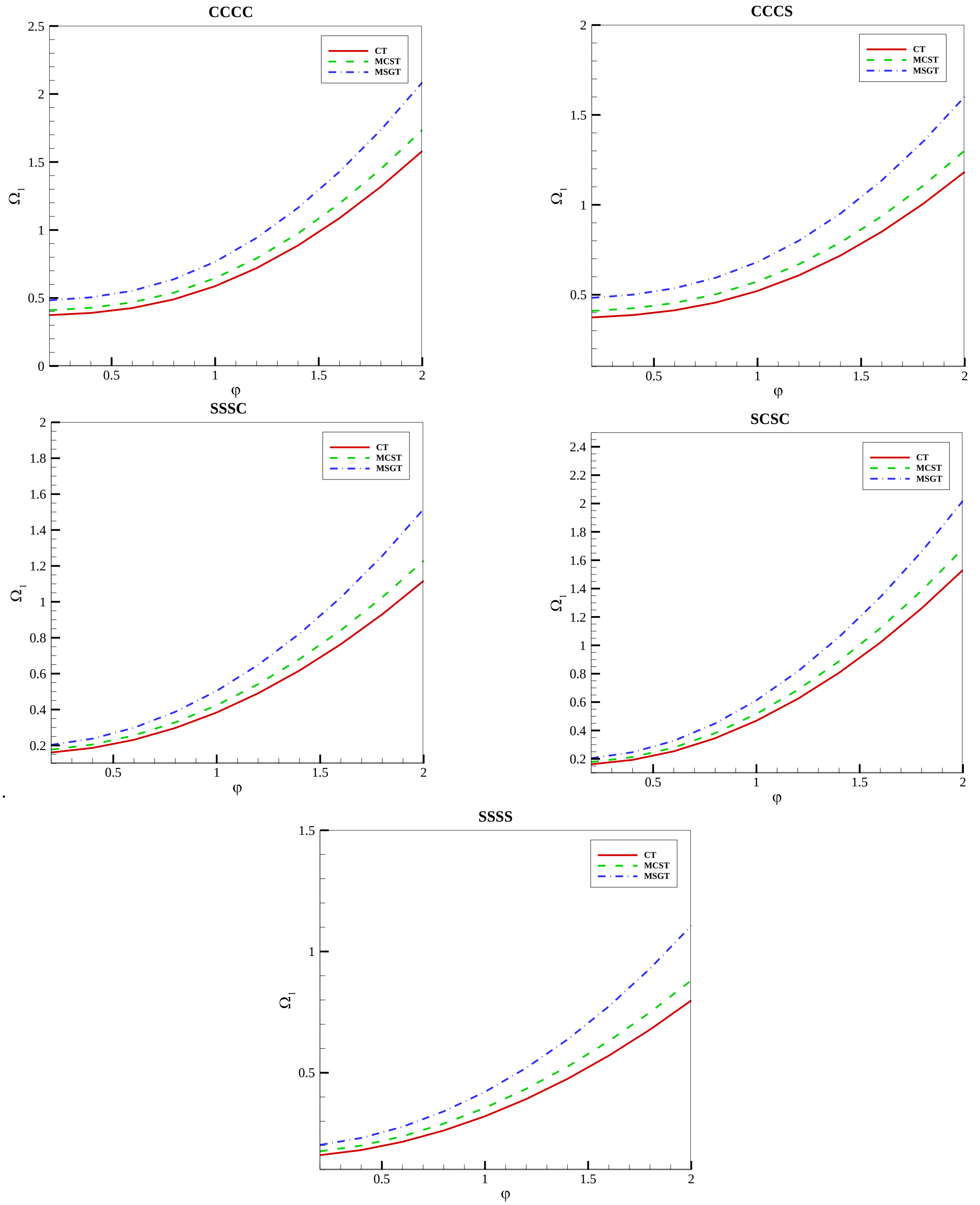


Fig. 11 The first dimensionless natural frequency of the FG squared nanoplate based on $\varphi = a/b$ on the basis of the MSGT, MCST and CT for $n = 2,\ \ h/l = 4,\ \beta_1 = 2,\ \beta_2 = 0$ and $h/a = 0.05$

It is clear from Fig. 7 that, irrespective of different boundary conditions, the gradient index of material (n) has different effects on the natural frequency. This result is due to the fact that the natural frequency is strongly dependent on the Young's modulus and density, and the gradient index of material has a very significant effect on the Young's modulus and density. It is also noteworthy that with decreasing the Young's modulus, the natural frequency decreases and with decreasing the density, the natural frequency increases. Consequently, for $0 < n < 1$, the density has a greater effect on vibrations compared to the Young's modulus. Therefore, by increasing the gradient index of material, the density decreases and the natural frequency increases. Conversely, for $n > 1$, the effect of Young's modulus increases. As a result, by increasing the gradient index of material, the Young's modulus decreases and the natural frequency decreases. To study the effects of the gradient index of material, (n), $\beta_1 = \beta_2 = 0, h/l = 4$ and $a/h = 20$ have been considered.

4.3 The effect of material gradient index β

Figs. 8 and 9 indicates the dimensionless frequency variations of the FG nanoplate based on β_1 and β_2 for three theories of classical, modified couple stress and modified strain gradient. As seen in the figure, the increase in the β causes the increasing of the dimensionless natural frequency for all boundary conditions. Increasing the β increases the hardness matrix, which also increases the natural frequency.

4.4 Effect of aspect ratio

Fig. 10 indicates the dimensionless frequency of the FG nanoplate based on the aspect ratio (ratio of length to thickness of nanoplate) for three classical, modified couple stress and modified strain gradient. As seen in the figure, the natural frequency decreases with increasing aspect ratio for all boundary conditions. Also, in Fig. 11, the effect of the length-to-width ratio of the plate is shown.

5. Conclusions

In this paper, the free vibrations of Kirchhoff's nanoplate made of a functionally graded material, is investigated based on the modified strain gradient theory. The properties of nanoplates change according to the arbitrary function in three directions of length, width and thickness of the plate. In addition to the modified strain gradient theory, numerical results are presented for two classical elasticity theory and modified couple stress theory, ignoring all or two length scale parameters respectively. The governing equations and boundary conditions are obtained based on Hamilton's Principle. Also, the generalized differential quadrature method is used to solve the governing equations for obtaining response of free vibrations of nanoplate. It was found that the effects of the size increase the hardness of the nanoplate and also increase the natural frequency of the FG nanoplate. It was also concluded that when the modified strain gradient theory is used as well as in harder boundary conditions, the effects of the size will become more prominent. It was also found that when the scale of structures becomes larger, small scale effects can be disregarded, and all three theories of strain gradient, couple stress, and classical give same answers. It was also found that when the aspect ratio increases, the natural frequency decreases. Also, to show the heterogeneity effects on the vibrations of FG nanoplate, different values of n and β were investigated. For $0 < n < 1$, the density possesses a greater effect on vibrations in comparison with the Young's modulus. Therefore, by increasing the gradient index of material, the density decreases and the natural frequency increases. Conversely, for $n > 1$, the effect of Young's modulus increases. As a result, by increasing the material gradient index, the Young's modulus decreases and also the natural frequency decreases. It has also been shown that with increasing β, the natural frequency increases.

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