Advances in Nano Research, *Vol. 8 No. 1 (2020) 13-24*
DOI: https://doi.org/10.12989/anr.2020.8.1.013

Torsional vibration analysis of bi-directional FG nano-cone with arbitrary cross-section based on nonlocal strain gradient elasticity

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(*Received January 8, 2019, Revised November 29, 2019, Accepted December, 17, 2019*)

Abstract. In this paper, for the first time based on the nonlocal strain gradient theory the effect of size dependency in torsional vibration of bi-direction functionally graded (FG) nonlinear nano-cone is study. The material properties were assumed to vary according to the arbitrary function in radial and axial directions. The Navier equation and boundary conditions of the size-dependent bidirectional FG nonlinear nano-cone were derived by Hamilton's principle. These equations were solved by employing the generalized differential quadrature method (GDQM). The presented model can turn into the classical model if the material length scale parameters are taken to be zero. The effects of some parameters, such as inhomogeneity constant, cross-sectional area parameter and small-scale parameters, were studied. As an essential result of this study can be stated that an FG nano-cone model based on the nonlocal elasticity theory behaves softer and based on the strain gradient theory behaves harder.

Keywords: torsional vibration; bi-directional FGMs; nano-cone; GDQM; nonlocal strain gradient elasticity

1. Introduction

Nanoscience is the important science of topic issue that are between in size the largest molecules and smallest dimension of the structures that can be obtained by common photolithography fabrication methods (Poole Jr and Owens 2003, Dowling 2004). Also, nanomaterials such as Functionally graded materials (FGMs) have numerous applications in general, nanotechnology and nanomechanics in particular because of their rare properties such as thermal and corrosive resistance. Hence, researchers in the last decade have unprecedentedly focused on using beams and plates made of FGMs to design, fabrication and characterization of nanomaterials with possible applications in enginee ring (Pompe *et al*. 2003, Miyamoto *et al*. 2013, Phung-Van *et al*. 2017c, Eltaher *et al*. 2018, Goyal and Soni 2018, Petit *et al*. 2018). The small structures such nanostructures are strongly attracting widespread attention of researchers since 1990s for many reasons and unknown properties of these nanostructures because of altering and reform these properties in the nanometer regime from bulk materials. The various application of these nanostructures is in different systems like biological and nanomechanical devices, fluid storage, fluid transport, drugs delivery tools and many disciplines such as biotechnology, medicine, chemical, and engineering. One of the first problems was that classical mechanics could not examine issues in the nano-scale. In order to solve this problem, reinforcement continuum mechanics theories have been proposed, which consider inherent characteristics of materials at the nano-scale. In recent decades, with the development of various engineering fields related to micro- and nano-systems, much attention has been given to size effects on material behaviors. Some advanced methods to address the weaknesses of the conventional theory are; Cosserat continuum mechanics (Kafadar and Eringen 1971a, b), nonlocal elasticity theory (Eringen 1972, 1983, 2002), strain gradient elasticity (Mindlin and Eshel 1968), couple stress theory (Toupin 1962) and nonlocal strain gradient theory (Lim *et al*. 2015). Some of the researchers by using of couple stress theory analyze the microstructure (Thanh *et al*. 2018, 2019a, b, c, d). The nonlocal elasticity theory predicts softening in nano size structures, but strain gradient theory indicates that the nanostructure experience hardening in nano scale than bulk materials. Recently, Lim *et al*. (2015) demonstrate the nonlocal strain gradient theory can predict both increase and decrease in the structural stiffness which is confirmed by the experimental data (Abazari *et al*. 2015). Many researchers today use nonlocal elasticity, strain gradient theory and nonlocal strain gradient theory to analyze of nanostructures (Asemi and Farajpour 2014, Farajpour *et al*. 2016, Hosseini *et al*. 2016, Nejad and Hadi 2016a, b, Nejad *et al*. 2016, Farajpour and Rastgoo 2017, Farajpour *et al*. 2017a, b, Hosseini *et al*. 2017, Phung-Van *et al*. 2017a, b, Rahmani *et al*. 2017, Shishesaz *et al*. 2017, Ebrahimi and Haghi 2018, Ebrahimi and Salari 2018, Farajpour *et al*. 2018a, b, c, Hadi *et al*. 2018a, b, Hosseini *et al*. 2018, 2019, Phung-Van *et al*. 2019). Fig. 1

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http://www.techno-press.org/?journal=journal=anr&subpage=5 ISSN: 2287-237X (Print), 2287-2388 (Online)

shows citation and the links of documents that were published on the nanoscience and were utilized nonlocal strain gradient theory since last years. Li *et al*. (2015) proposed a nano- scale functionally graded (FG) beam analytical model for flexural wave propagation based on the theory of non- local strain gradients. They provided the explicit acoustical and optical dispersion relations between phase velocity and wave number. Hadi *et al*. (2018a) explored the free vibration of three-directional FGMs in Euler–Bernoulli nano-beam. They employed the nonlocal strain gradient theory to study the small-scale effects to exploring the free vibration of the three – directional FG Euler–Bernoulli nano-beam. These authors illustrated that in nonlocal strain gradient theory, the natural frequency could either be higher than the natural frequency of classical theory or smaller. Karami *et al*. (2018a-d) utilized the second-order shear deformation concept and nonlocal strain gradient theory to investigate the initial stress influenced wave dispersion in graphene. They derived constitutive orthotropic equations based on the nonlocal strain gradient and created an estimate subordinate 2D continuum model to supply a conceivable hypothetical way to deal with researching the wave practices of Graphene-like 2D materials.

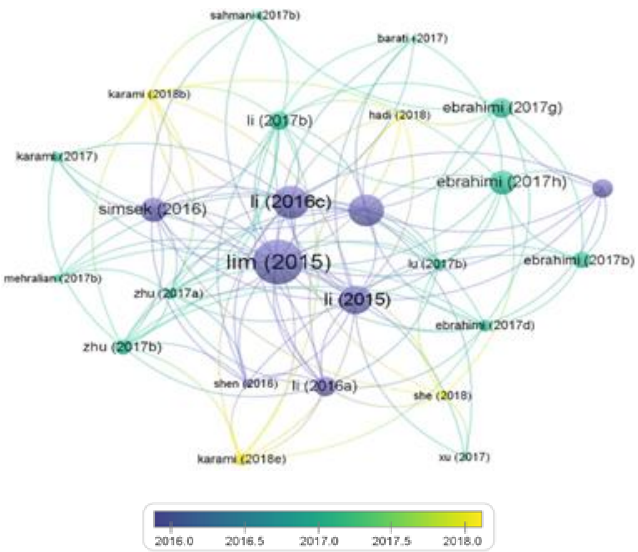

Fig. 1 Citation and the links of documents that were published and utilized nonlocal strain gradient theory since last years. The links between the two documents demonstrate that one document is cited from the other. The knot size shows the number of documents cited globally and the color bar indicates the year in which the paper is published. (Lim *et al*. 2015, Li and Hu 2016, Li *et al*. 2016, Shen *et al*. 2016, Simsek 2016, Barati 2017, Ebrahimi and Barati 2017a, b, Ebrahimi *et al*. 2017a, b, Ebrahimi and Dabbagh 2017, Ebrahimi and Haghi 2017, Li and Hu 2017, Lu *et al*. 2017a, b, Mehralian *et al*. 2017a, b, Sahmani and Aghdam 2017, Xu *et al*. 2017, Zhu and Li 2017a, b, Hadi *et al*. 2018a, Karami *et al*. 2018a-d, She *et al*. 2018)

Functionally Graded Materials (FGMs) depicted by the variety in blend and arrangement step by step over volume, bringing about relating changes in the properties of the material (Zamani Nejad *et al*. 2017, Akbas 2018, Aydogdu *et al*. 2018, Ebrahimi and Barati 2018a, b, Ebrahimi and Fardshad 2018, Nejad *et al*. 2018, Noroozi and Ataee 2018, Thai *et al*. 2018, Bodaghi *et al*. 2019). FG materials not only decrease the weight but also increment the strength. Inside the most recent two years, investigators employed nonlocal strain gradient theory to simulate vibration analysis of FGMs (Ebrahimi and Barati 2018a, b). Wave spread investigation of rotating thermoelastically-actuated nanobeams dependent on nonlocal strain gradient theory was studied by Ebrahimi and Haghi (2017). Their outcomes illuminate that different parameter like temperature change, precise speed, nonlocality parameter, nonlocality parameter, and gradient index impact on the wave scattering of the nano-beam.

Additionally, Ebrahimi and Haghi (2018) published the next paper on a nonlocal strain gradient theory for scale-subordinate wave scattering examination of turning FG nano-beams. In this paper, they considered the physical field impacts on the wave scattering qualities of nanobeam. They researched the wave proliferation conduct of rotating functionally graded temperature-dependent nanoscale beams subjected to thermal loading based on the nonlocal strain gradient stress field. Besides, nonlinear temperature conveyances over the thickness were examined by these authors. Li *et al*. (2018) investigated the nonlinear vibration investigation of nano-beam made of porous materials. They utilized Hamilton’s principle to get the size-subordinate nonlinear conditions of movement dependent on the Euler–Bernoulli beam and the nonlocal strain gradient elasticity theory.

Only a few researchers worked on torsional vibration based on nonlocal strain gradient elasticity theory. Fig. 2 shows the documents and their links published on the torsional vibration of nanoscience based on nonlocal elasticity theory. Li (2014) compared two nonlocal models and a semi-continuum model for investigating the torsional vibration of carbon nanotubes. Adeli *et al*. (2017) explored free torsional vibration conduct of a nonlinear nano-

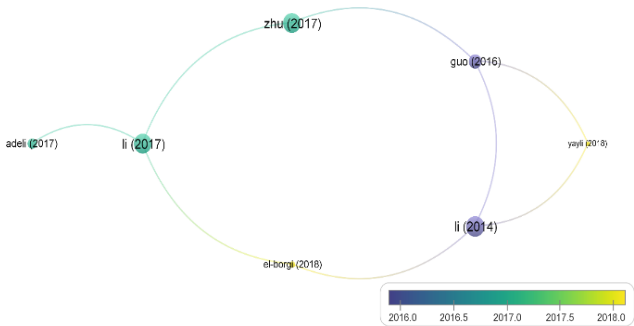

Fig. 2 Citation and the links of documents that investigated torsional vibration behavior by nonlocal strain gradient theory since last years (Li 2014, Guo *et al*. 2016, Adeli *et al*. 2017, Li and Hu 2017, Zhu and Li 2017a, b, El-Borgi *et al*. 2018, Yayli 2018)

cone, in light of the nonlocal strain gradient theory. The impacts of a few parameters, for example, cross-sectional zone change and little scale parameter, were examined in their paper. Results demonstrate that the cross-sectional zone change significantly affects the torsional vibration conduct of the nano-cone. Li and Hu (2017) introduced the condition of torsional movement to explore the free torsional vibration practices of tubes made of a bi-directional functionally graded (FG) material. They determined the closed-form solutions of torsional frequencies and mode shapes. They demonstrated that the torsional frequencies could be essentially influenced by the through-radius and through-length grading of the bidirectional FG nanotubes and thus can be recommended by fitting the bidirectional nano-structures of the FG material. They have shown that the torsional frequencies can be expanded with the diminishing nonlocal parameter, while the size-subordinate practices on the mode shape cannot be watched. Zhu and Li (2017a, b) solved the longitudinal problem of a size-subordinate rod by using an integral form of nonlocal strain gradient theory. In their investigation, it was demonstrated unequivocally that the integral rod model could apply stiffness-softening and stiffness-hardening impacts by thinking about different estimations of the size-subordinate parameters. Moreover, by concentrate, the size-subordinate consequences for the longitudinal elements of monolayer graphene, the scattering connection determined by utilizing the nonlocal strain gradient model can demonstrate great concurrence with the test information gotten by inelastic X-ray dissipating (Zhu and Li 2017b).

In this paper, to the best of the researchers' knowledge, for the first time, using nonlocal strain gradient theory, the torsional vibration of bi-directional nano-cone is investigated. The effects of changes of some crucial parameters such as material length scale, FG index on the values of torsional frequencies are studied. The results of this study can be a reference for designing the nano-size devices.

2. Analysis

Consider a nano-cone made of bi-directional functionally graded material with length L, Radius R_i, and R_o in $z = 0$ and $z = L$, respectively. The radius of cross-section graded along z-direction with arbitrary nonlinear function. The schematic of the nano-cone is shown in Fig. 3.

The material properties of the nano-size cone are assumed to vary according to the arbitrary function along z and r directions.

$$G(r,z) = G_1(r)G_2(z) \tag{1}$$

$$\rho(r,z) = \rho_1(r)\rho_2(z) \tag{2}$$

where G and ρ denote the shear modulus and density, respectively.

The nonlocal strain gradient can predict all of the material properties by considering softening and hardening

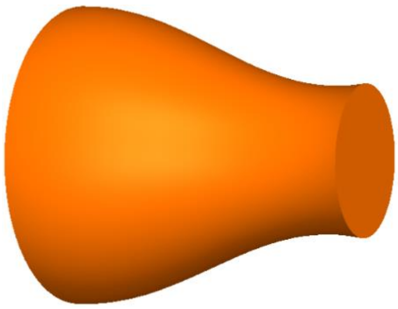

Fig. 3 The geometry of nano-cone with arbitrary section

in nanosize structures. The constitutive equation for the behavior of the material based on nonlocal strain gradient theory is defined as follow

$$[1-\mu^2\nabla^2]\begin{Bmatrix} t_{xz} \\ t_{yz} \end{Bmatrix} = 2G(1-l^2\nabla^2)\begin{Bmatrix} \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix} \tag{3}$$

in Eq. (3) and (4), ∇ is the Nabla operator. μ and l are size-dependent parameters for nonlocal and strain gradient theory. ε_{ij} denote the shear strain. Moreover, the total stress (t_{ij}) is defined as

$$t_{ij} = \sigma_{ij} - \nabla\sigma_{ijm}^{(1)} \tag{4}$$

where, σ_{ij} and $\sigma_{ijm}^{(1)}$ are stress and higher-order stress.

The displacement field for torsion can be expressed as

$$\begin{cases} u = -y\phi \\ v = x\phi \quad ,\phi = \phi(z,t) \\ w = 0 \end{cases} \tag{5}$$

where u, v, and $w$are components of the displacement along x, y, and z-direction and t denotes time. Considering small deformation, non-zero components of strain tensor is expressed as follow

$$\varepsilon_{xz} = \varepsilon_{zx} = -\frac{y}{2}\frac{\partial\phi}{\partial z} \tag{6}$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{x}{2}\frac{\partial\phi}{\partial z} \tag{7}$$

That ϕ shows the torsion field along the z-direction. The governing motion equation of bi-directional FG nano-cone is derived using Hamilton's principle as

$$\int_{t_1}^{t_2}(\delta U - \delta K)\,dt = 0 \tag{8}$$

δU and, δK are variation of potential and kinetic energy, respectively.

The variation of potential energy-based nonlocal strain gradient theory for torsional vibration can be described as follow

$$\delta U = \int_V \begin{pmatrix} 2\sigma_{xz}\delta\varepsilon_{xz} + 2\sigma_{yz}\delta\varepsilon_{yz} + \\ 2\sigma^{(1)}{}_{xzz}\nabla\delta\varepsilon_{xz} + 2\sigma^{(1)}{}_{yzz}\nabla\delta\varepsilon_{yz} \end{pmatrix} dV \tag{9}$$

where $\nabla\delta\varepsilon_{ij}$ are first-order strain gradient. By simplifying and integral by part, the potential energy is defined.

$$\delta U = -\int_0^L \frac{\partial T_z}{\partial z}\delta\phi dz + \left[T_z\delta\phi + T_Z{}^{(1)}\delta\frac{\partial\phi}{\partial Z}\right]_0^L \qquad (10)$$

in the above equation, T_z, and $T_Z{}^{(1)}$ describe as follow

$$T_z = \int_A (xt_{yz} - yt_{xz})\, dA \qquad (11)$$

$$T_z{}^{(1)} = \int_A (x\sigma^{(1)}{}_{yzz} - y\sigma^{(1)}{}_{xzz})\, dA \qquad (12)$$

The variation of the kinetic energy is given by

$$\delta K = \int_0^t \int_V \rho[\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w}]\, dVdt \qquad (13)$$

Substituting Eqs. (1), (2), and (5) into Eq. (12) and using integral by part the variation of the kinetic energy derived.

$$\begin{aligned}\delta K &= \int_0^t \int_0^L \rho_2(z)J_o\,\dot{\phi}\delta\dot{\phi}dzdt \\ &= \int_0^L \rho_1(z)J_o\frac{\partial\phi}{\partial t}\delta\phi dz|_0^t \\ &\quad -\int_0^t\int_0^L \rho_1(z)J_o\frac{\partial^2\phi}{\partial t^2}\delta\phi dzdt\end{aligned} \qquad (14)$$

where the polar moments of mass

$$J_o = \int \rho_1(r)\, r^2 dA \qquad (15)$$

Substituting Eqs. (10), and (15) into Eq. (8), the governing equation expressed as follow

$$\frac{\partial T_z}{\partial z} - \rho_2(z)J\frac{\partial^2\phi}{\partial t^2} = 0 \qquad (16)$$

The classical boundary condition is

$$\phi = 0, or T_z = 0 \qquad (17)$$

and non-classical boundary condition is defined as follow

$$\frac{\partial\phi}{\partial z} = 0, or T_z{}^{(1)} = 0 \qquad (18)$$

Substituting Eqs. (6)-(7), (11) into Eq. (16), the equation of torsional motion for the FG nano-cone with arbitrary cross-section can be expressed as follow

$$\begin{aligned}&\rho_2(z)J_o\frac{\partial^2\phi}{\partial t^2} - \mu^2\left(\begin{matrix}J_o\dfrac{\partial^2\rho_2(z)}{\partial z^2} + 2\dfrac{\partial\rho_2(z)}{\partial z}\dfrac{\partial J_o}{\partial z} + \\ \rho_1(z)\dfrac{\partial^2 J_o}{\partial z^2}\end{matrix}\right)\frac{\partial^2\phi}{\partial t^2} \\ &-2\mu^2\left(\begin{matrix}\dfrac{\partial\rho_2(z)}{\partial z}J_o + \\ \rho_2(z)\dfrac{\partial J_o}{\partial z}\end{matrix}\right)\frac{\partial^3\phi}{\partial z\partial t^2} - \mu^2\rho_2(z)J_o\frac{\partial^4\phi}{\partial t^2\partial z^2}) \\ &= \frac{\partial G_2(z)}{\partial z}(\frac{\partial\phi}{\partial z} - l^2\frac{\partial^3\phi}{\partial z^3})I_o + G_2(z)(\frac{\partial^2\phi}{\partial z^2} - l^2\frac{\partial^4\phi}{\partial z^4})I_o \\ &+G_2(z)(\frac{\partial\phi}{\partial z} - l^2\frac{\partial^3\phi}{\partial z^3})\frac{\partial I_o}{\partial z}\end{aligned} \qquad (19)$$

In free vibration, ϕ can be decomposed in two-part, that φ varies harmonically with respect to the time variable t

$$\phi(z,t) = \varphi(z)e^{i\omega t} \qquad (20)$$

By substituting Eq. (20) into Eq. (19), Eq. (19) can be written as

$$\begin{aligned}&\left(\begin{matrix}-\rho_2(z)J_o\varphi + \\ \mu^2\left(\begin{matrix}\left(\begin{matrix}J_o\dfrac{\partial^2\rho_2(z)}{\partial z^2} + 2\dfrac{\partial\rho_2(z)}{\partial z}\dfrac{\partial J_o}{\partial z} + \\ \rho_2(z)\dfrac{\partial^2 J_o}{\partial z^2}\end{matrix}\right)\varphi \\ +2\left(\dfrac{\partial\rho_2(z)}{\partial z}J_o + \rho_2(z)\dfrac{\partial J_o}{\partial z}\right)\dfrac{\partial\varphi}{\partial z} + \\ \rho_2(z)J_o\dfrac{\partial^2\varphi}{\partial z^2}\end{matrix}\right)\end{matrix}\right)\omega^2 \\ &= \frac{\partial G_2(z)}{\partial z}\left(\frac{\partial\varphi}{\partial z} - l^2\frac{\partial^3\varphi}{\partial z^3}\right)I_o + G_2(z)(\frac{\partial^2\varphi}{\partial z^2} - l^2\frac{\partial^4\varphi}{\partial z^4})I_o \\ &+G_2(z)(\frac{\partial\varphi}{\partial z} - l^2\frac{\partial^3\varphi}{\partial z^3})\frac{\partial I_o}{\partial z}\end{aligned} \qquad (21)$$

$$\frac{\partial\phi}{\partial z} = 0, or T_z{}^{(1)} = 0 \qquad (22)$$

In order to obtain general results, the following dimensionless quantities can be defined

$$\begin{aligned}&\Omega^2 = \frac{J_o\omega^2 L}{I_o} \quad \bar{z} = \frac{z}{L} \\ &\bar{\mu} = \frac{\mu}{L} \quad \bar{l} = \frac{l}{L} \quad \bar{R} = \frac{R}{L}\end{aligned} \qquad (23)$$

That Ω , $\bar{z}$, $\bar{\mu}$, $\bar{l}$, $\bar{R}$ shows the nondimensional frequency, longitudinal component, size-dependent parameters for nonlocal and strain gradient theory and radius, respectively.

The equation of motion bi-directional FG nano-cone can be written in the dimensionless form as follows

$$\begin{aligned}&\left(\begin{matrix}-\rho_2(\bar{z})\varphi + \bar{\mu}^2\dfrac{\partial^2\rho_2(\bar{z})}{\partial\bar{z}^2}\varphi + 2\bar{\mu}^2\dfrac{\partial\rho_2(\bar{z})}{\partial\bar{z}}\dfrac{\partial J_o}{\partial\bar{z}}\dfrac{1}{J_o}\varphi \\ +\bar{\mu}^2\rho_2(\bar{z})\dfrac{\partial^2 J_o}{\partial\bar{z}^2}\varphi + 2\bar{\mu}^2\dfrac{\partial\rho_2(\bar{z})}{\partial\bar{z}}\dfrac{\partial\varphi}{\partial\bar{z}} + \\ 2\bar{\mu}^2\rho_2(\bar{z})\dfrac{\partial J_o}{\partial\bar{z}}\dfrac{1}{J_o}\dfrac{\partial\varphi}{\partial\bar{z}} + \bar{\mu}^2\rho_2(\bar{z})\dfrac{\partial^2\varphi}{\partial\bar{z}^2}\end{matrix}\right)\Omega^2 \\ &= \frac{\partial G_2(\bar{z})}{\partial\bar{z}}(\frac{\partial^2\varphi}{\partial\bar{z}^2} - \bar{l}^2\frac{\partial^3\varphi}{\partial\bar{z}^3})I_o + G_2(\bar{z})(\frac{\partial^2\varphi}{\partial\bar{z}^2} - \bar{l}^2\frac{\partial^4\varphi}{\partial\bar{z}^4})I_o \\ &+G_2(\bar{z})(\frac{\partial\varphi}{\partial\bar{z}} - \bar{l}^2\frac{\partial^3\varphi}{\partial\bar{z}^3})\frac{1}{I_o}\frac{\partial I_o}{\partial\bar{z}}\end{aligned} \qquad (24)$$

3. Generalized differential quadrature method

In the case of the general boundary conditions, the analytical solution of Eq. (22) is difficult to obtain, so a generalized differential quadrature (GDQ) approach has been undertaken for the solution of Eq. (22). The GDQ approach may be an easy and useful tool for analyzing problems that are more complex. The generalized differential quadrature method is an efficient numerical method for the solution of differential equations. It is assumed that the grid points are located on the zeros of the Chebyshev polynomials (Shu and Chew 1998) and to discretize the solution domain, one can assume a set of N grid points in the x -direction

$$Z_i = \frac{L}{2}\left\{1 - cos\left(\frac{i-1}{N-1}\right)\pi\right\}, \qquad i = 1, \dots, N-1 \tag{25}$$

In this method, the derivatives of a function, at a point z_i are expressed as

$$\varphi_x^{(n)}(z_i) = \sum_{j=1}^{N} C_{ij}^{(n)} \varphi(z_j), \qquad n = 1, \dots, N-1 \tag{26}$$

where N is the numbers of the grid points over the x direction. $C_{ij}^{(n)}$ is the respective weighting coefficients through the z direction obtained according to the following equations.

If $n = 1$, i.e., for the first-order derivative, then

$$C_{ij}^{(1)} = \frac{M(Z_i)}{(Z_i - Z_j)M(Z_j)}, \qquad i,j = 1, \dots, N \quad j \neq i \tag{27}$$

Where

$$M(Z_i) = \prod_{\substack{j=1 \\ j\neq i}}^{N_x} (Z_i - Z_j) \tag{28}$$

To obtain the weighting coefficients for the second-order or higher-order derivatives, the matrix multiplication procedure is implemented

$$C_{ij}^{(n)} = n\left(C_{ii}^{(n-1)} C_{ij}^{(1)} - \frac{C_{ij}^{(n-1)}}{Z_i - Z_j}\right), \quad i,j = 1, \dots, N \quad j \neq i \tag{29}$$

$$C_{ii}^{(n)} = -\sum_{\substack{j=1 \\ j\neq i}}^{N_x} C_{ij}^{(n)}, \quad \begin{cases} i = 1, \dots, N \\ n = 1,2, \dots, N-1 \end{cases} \tag{30}$$

Substituting Eq. (25) into the first governing equation (Eq. (24)) results in the following equation

$$\begin{aligned}
&\Bigg[-\rho_2(z_i)\sum_{j=1}^{N} C_{ij}\varphi(z_j) + \bar{\mu}^2 \frac{\partial^2\rho_2(\bar{z})}{\partial\bar{z}^2}\bigg|_{z=z_i} \sum_{j=1}^{N} C_{ij}\varphi(z_j) \\
&+2\bar{\mu}^2\left(\frac{\partial\rho_2(\bar{z})}{\partial\bar{z}}\frac{\partial J_o}{\partial\bar{z}}\frac{1}{J_o}\right)\bigg|_{z=z_i} \sum_{j=1}^{N} C_{ij}\varphi(\bar{z}_j) + \bar{\mu}^2\rho_2(\bar{z}_i)\frac{\partial^2 J_o}{\partial\bar{z}^2}\bigg|_{z=z_i} \\
&\sum_{j=1}^{N} C_{ij}\varphi(\bar{z}_j) + 2\bar{\mu}^2\frac{\partial\rho_2(\bar{z})}{\partial\bar{z}}\bigg|_{z=z_i} \sum_{j=1}^{N} C_{ij}^{(1)}\varphi(\bar{z}_j) + 2\bar{\mu}^2\rho_2(\bar{z}_i) \\
&\left(\frac{\partial J_o}{\partial\bar{z}}\frac{1}{J_o}\right)\bigg|_{\bar{z}=\bar{z}_i} \sum_{j=1}^{N} C_{ij}^{(1)}\varphi(\bar{z}_j) + \bar{\mu}^2\rho_2(\bar{z}_i)\sum_{j=1}^{N} C_{ij}^{(2)}\varphi(\bar{z}_j)\Bigg]\Omega^2 \\
&= \frac{\partial G_2(\bar{z})}{\partial\bar{z}}\bigg|_{\bar{z}=\bar{z}_i}\left(\sum_{j=1}^{N} C_{ij}^{(2)}\varphi(\bar{z}_j) - \bar{l}^2\sum_{j=1}^{N} C_{ij}^{(3)}\varphi(\bar{z}_j)\right) \\
&+G_2(\bar{z}_i)\left(\sum_{j=1}^{N} C_{ij}^{(1)}\varphi(\bar{z}_j) - \bar{l}^2\sum_{j=1}^{N} C_{ij}^{(3)}\varphi(\bar{z}_j)\right)\left(\frac{1}{I_o}\frac{\partial I_o}{\partial\bar{z}}\right)\bigg|_{\bar{z}=\bar{z}_i} \\
&+G_2(\bar{z}_i)\left(\sum_{j=1}^{N} C_{ij}^{(2)}\varphi(\bar{z}_j) - \bar{l}^2\sum_{j=1}^{N} C_{ij}^{(4)}\varphi(\bar{z}_j)\right)
\end{aligned} \tag{31}$$

Arranging the displacement variable and corresponding coefficient, the governing equations in the following form can be obtained

$$\begin{bmatrix} A_{bb} & A_{bd} \\ A_{db} & A_{dd} \end{bmatrix}\begin{bmatrix} X_b \\ X_d \end{bmatrix} = [\Omega]\begin{bmatrix} 0 & 0 \\ B_{db} & B_{dd} \end{bmatrix}\begin{bmatrix} X_b \\ X_d \end{bmatrix} \tag{32}$$

In which subscripts b and d denote boundary and domain sample points, respectively. In addition, coefficients A and B are matrices and their dimensions depend on the number of domain and boundary sample points. After eliminating boundary nodes X_b in Eq. (31) by using the boundary conditions, the dimension of the coefficient matrices reduces. Finally, Eq. (31) can be rewritten to give an eigenvalue problem as

$$[K][X_d] = [\,\Omega][I][X_d] \tag{33}$$

Solving the obtained eigenvalue problem gives the natural torsional frequency (Ω) for the bi-directional FG nano-cone based on nonlocal strain gradient theory.

3. Result and discussion

In this section, we consider bi-directional FG nano-cone that its radius changes along z with Eq. (34). Fig. 4 shows the change cross-section for different quantities n. The properties of nano-cone are different along z and r, and we can decompose mechanical properties in the form of product longitudinal term and radius term. Eq. (33) expressed decompose of mechanical properties, that η in an arbitrary property. The longitudinal properties FG nano-cone is varying with an exponential function that expressed with Eqs. (34) and (35), that show different shear modulus and density, respectively. Fig. 5, shows distribution density and shear modules along the z-direction. The radial properties are varying with the power function that expressed with Eqs. (36) and (37). Fig. 6, shows distribution density and shear modules along r direction.

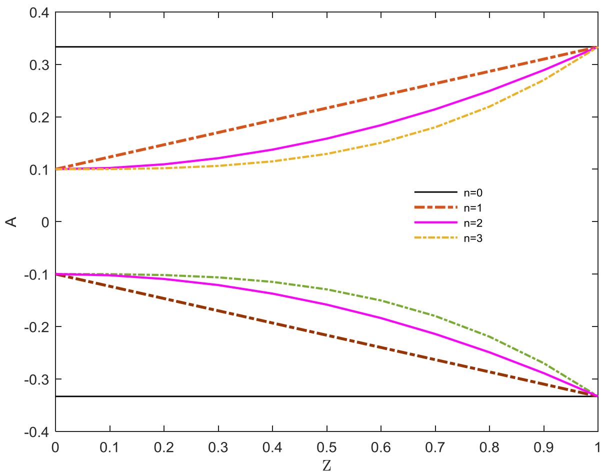

Fig. 4 cross-section variations for different values of n

$$R(z) = (R_o - R_i)\left(\frac{z}{L}\right)^n + R_i \tag{34}$$

$$\eta = \eta_1(r)\eta_2(z) \tag{35}$$

where

$$G_2(z) = G_o e^{(m_1 \frac{z}{L})} \tag{36}$$

$$\rho_2(z) = \rho_o e^{(m_2 \frac{z}{L})} \tag{37}$$

$$\frac{\partial \phi}{\partial z} = 0, or T_z^{(1)} = 0 \tag{38}$$

$$G_1(r) = (1 + \frac{r}{Ro})^{\beta_1} \tag{39}$$

$$\rho_2(r) = (1 + \frac{r}{Ro})^{\beta_2} \tag{40}$$

In order to verify the present work, a comparison frequency is made with Demir and Civalek (2013). For the comparison, we should be neglecting the strain gradient material length scale parameter ($l = 0$) and consider a nano-cone with a constant cross-section ($n = 0$). Furthermore, this comparison is the dimensionless frequency in mode one of clamped-clamped support (C-C) nano-cone with various nonlocal material length scale parameter, as shown in Table 1. As observed, the results of this paper are consistent with those of Demir and Civalek (2013).

In this section, we study the effect of different bi-directional FG nano-cone parameters $(m_1, m_2, \beta_1, \beta_2)$ and different cross-section parameters (n) on torsional vibration. At first, we check the number of the discrete points at DGQ and its influence on frequency. Fig. 7 shows convergence the nondimensional frequency obtained from DGQ for different quantities n. This result shows that for the different cross section, by choosing $N = 15$, we can be sure that convergence has happened $(error = 10^{-3})$.

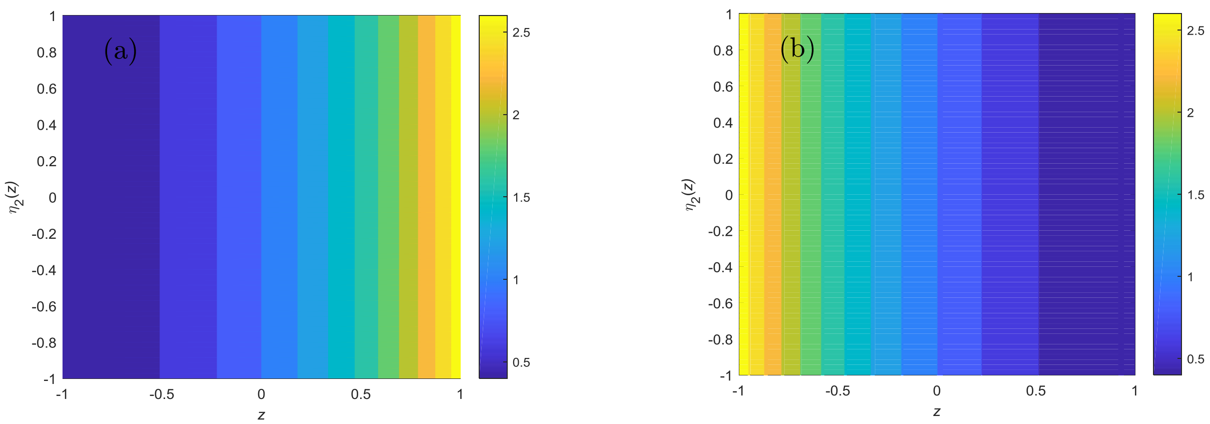

Fig. 5 Longitudinal properties FG nano-cone along z for (a) $m = 1$; (b) $m = -1$

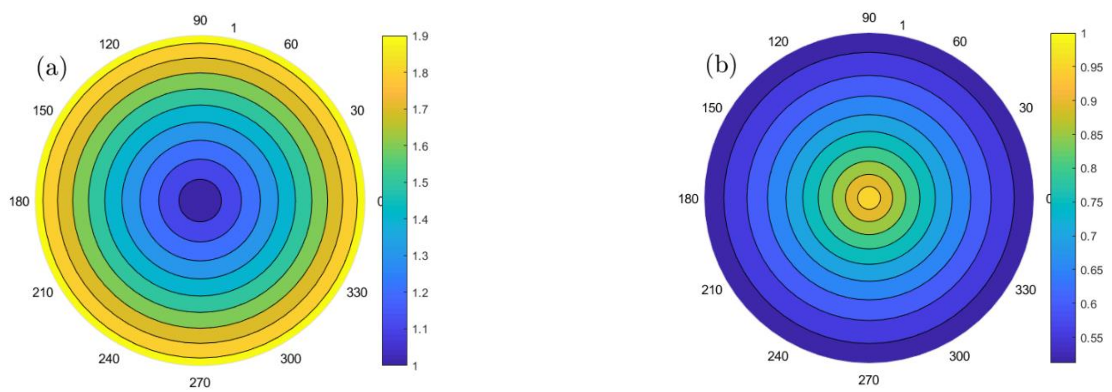

Fig. 6 radial properties FG nano-cone along r for (a) $\beta = 3$; (b) $\beta = -3$

Table 1 Comparison non-dimensional frequency obtained from percent work with (Demir and Civalek 2013)

Mode number	$\bar{\mu}$	**Ω** [Present work]	**Ω** (Demir and Civalek 2013)
1		3.1416	3.1416
2		6.2832	6.2832
3	0	9.4248	9.4248
4		12.5665	12.5664
5		15.7097	15.7080
1		3.1035	3.1035
2		5.9943	5.9943
3	0.5	8.5256	8.5256
4		10.6405	10.6404
5		12.3542	12.3534
1		2.9972	2.9972
2		5.3202	5.3202
3	0.1	6.8587	6.8587
4		7.8248	7.8248
5		8.4359	8.4356
1		2.8419	2.8419
2		4.5724	4.5724
3	0.15	5.4427	5.4427
4		5.8892	5.8892
5		6.1369	6.1368

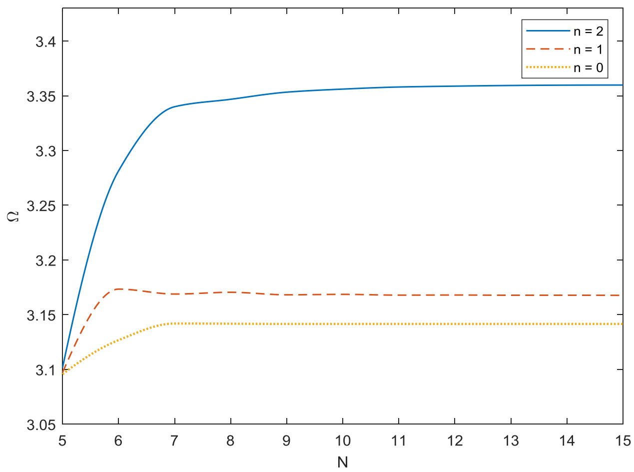

Fig. 7 The convergence of dimensionless torsional frequency in mode 1 clamped-clamped FG nano-cone for $m_1 = m_2 = \beta_1 = \beta_2 = 0$ and $l = \mu = 1\,nm$

Fig. 8 shows the ratio of natural frequency in strain gradient theory than the natural frequency in the classical theory versus FG nano-cone length. As is shown in this figure, the natural frequency obtained from the strain gradient theory is larger than the classical theory. By increasing l, the effects of the size are increased and the difference between strain gradient theory and classical theory increasing, in other words, the material gets harder.

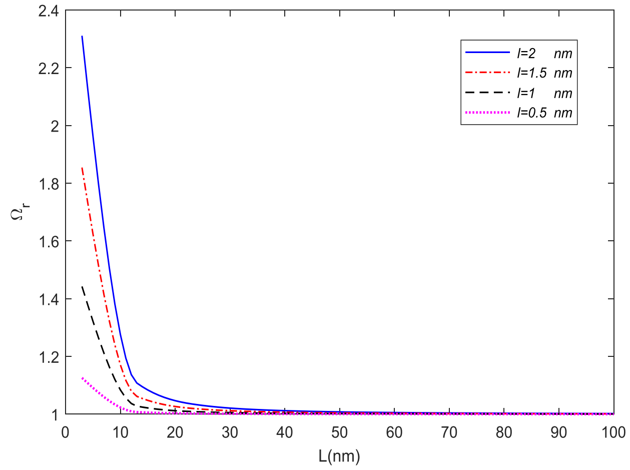

Fig. 8 Torsional frequency ratio of the clamped-clamped nano-cone versus length for $m_1 = \beta_1 = \beta_2 = \mu = 0$ and $m_2 = 1$, and $n = 2$

In very small length, this ratio is tremendous and with increasing length, this ratio decreasing. The trend continues until at L = 50 nm that in this length strain gradient and classical theory reports similar results, in another word the effects of the size disappear.

Fig. 9 shows the ratio of natural frequency in nonlocal elasticity theory to the natural frequency in the classical theory versus FG nano-cone length for different cross-section parameters (n). In contrast to the staring radiant theory, the natural frequency obtained from the nonlocal theory is smaller than the classical theory. By increasing μ, the effects of the size increased, and the difference between nonlocal elasticity theory and classical theory is increasing.

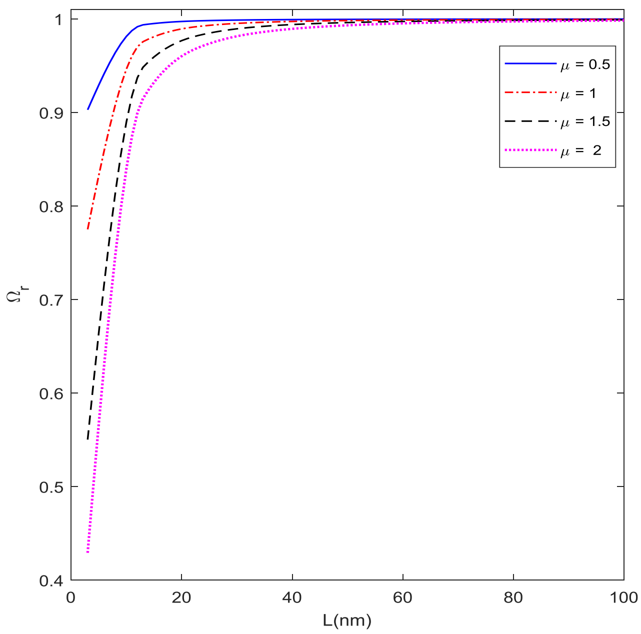

Fig. 9 Torsional frequency ratio of the clamped-clamped nano-cone versus length for $\beta_1 = \beta_2 = l = 0$ and $m_1 = m_2 = 1$, and $n = 2$

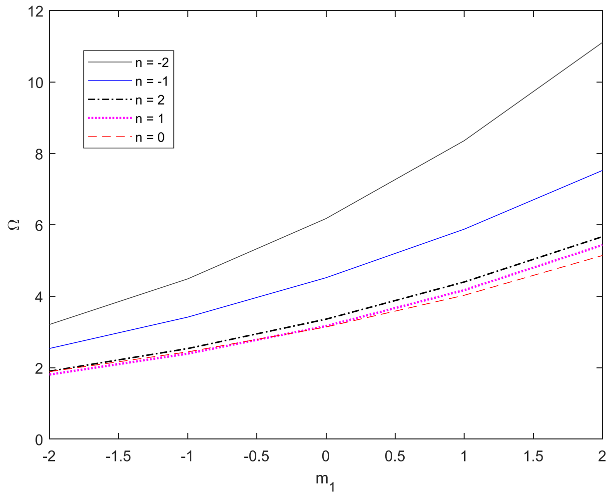

Fig. 10 Torsional frequency FG nano-cone versus m_1 length for $\beta_1 = \beta_2 = m_2 = 0$ and $l = \mu = 1\ nm$

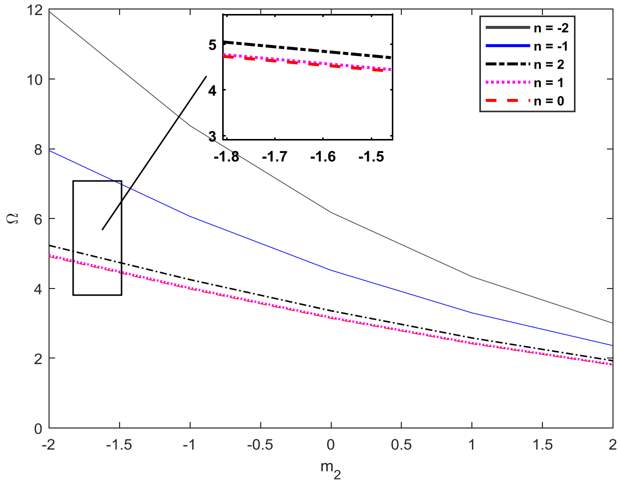

Fig. 11 Length for $\beta_1 = \beta_2 = m_1 = 0$ and Torsional frequency FG nano-cone versus $m_2 l = \mu = 1\ nm$

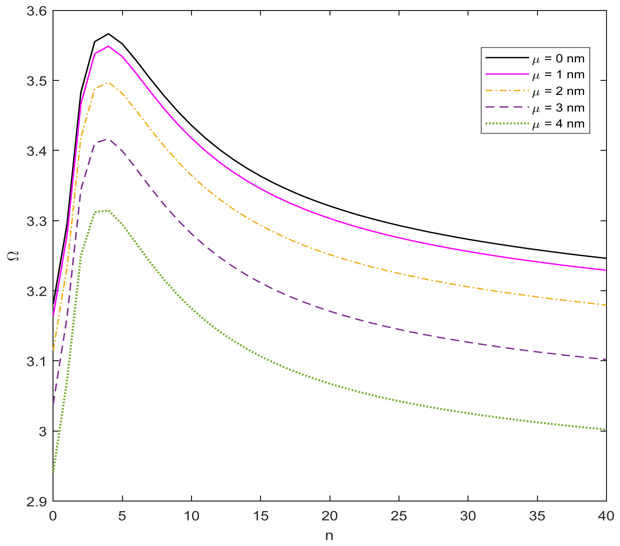

Fig. 12 Torsional frequency FG nano-cone versus n length for $\beta_1 = \beta_2 = l = 0$ and $m_1 = m_2 = 1$

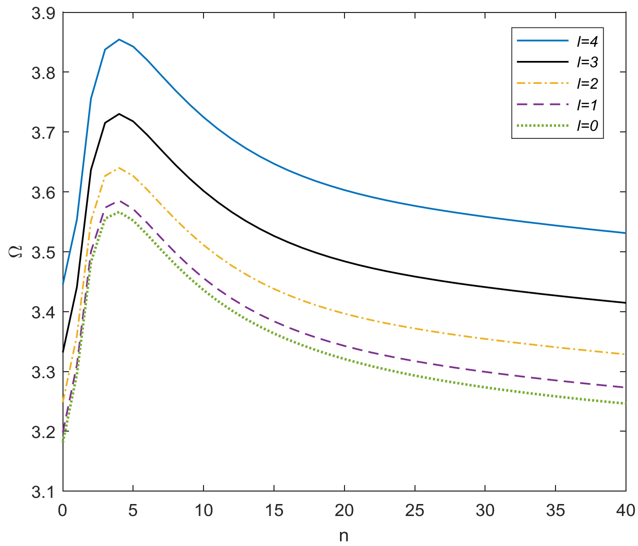

Fig. 13 Torsional frequency FG nano-cone versus $n$length for $\beta_1 = \beta_2 = \mu = 0$ and $m_1 = m_2 = 1$

In this theory, the material becomes softer. In petite length, this ratio is very small, and with increasing length, this ratio is increasing. This trend continues until at L = 50 nm that in this length, the nonlocal and the classical theory is approach to each other.

Figs. 10 and 11 demonstrate the effect of m_1 and m_2 on the torsional vibration natural frequency, respectively. By increasing m_1 at a constant section, the torsional vibration frequency increased, which its reason is the hardening of the material. In contrast m_1, with increasing m_2, the torsional frequency decreased, in other words, by increasing density, the torsional frequency decreased.

Figs. 12 and 13 show torsional vibration frequency versus n for different quantities μ and l, respectively. In the constant μ and l, by increasing n, at first, the frequency increasing, this trend continues until $n = 5$, in this point, frequency reaches its maximum, after this point by increasing n, frequency decreasing. By increasing μ, the torsional frequency is decreasing, but in contrast μ, by increasing l, the torsional frequency increases.

Figs. 14-15 show the torsional vibration frequency versus n at different quantities m_1 and m_2, respectively. At first, for $m_1 < 0$, by increasing n, frequency is decreasing. This trend continues until $n = 1$, in this point, frequency reaches its minimum point. After $n = 1$, by increasing n, frequency increases. For large n, the cross-section change not a significant parameter on the natural frequency. In contrast the negative value of m_1, for $m_1 > 0$, As n increases, the frequency increases first and then decreases. In the positive and negative value ofm_1 and m_2, for large value of n, by increasing n, there is no effect on the frequency.

Figs. 16-17 showed the effect of β_1 and β_2 on the torsional vibration frequency at different quantities m_1. By increasing β_1, the frequency is increasing. In contrast β_1, by increasing β_2, the frequency decreases.

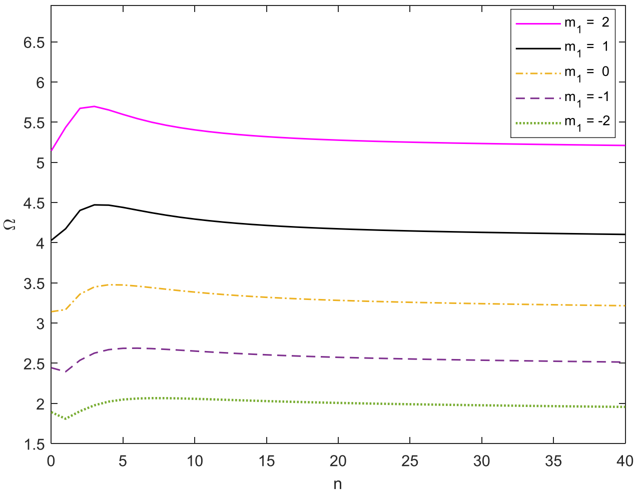

Fig. 14 Torsional frequency FG nano-cone versus n length for $\beta_1 = \beta_2 = m_2 = 0$ and $l = \mu = 1\,nm$

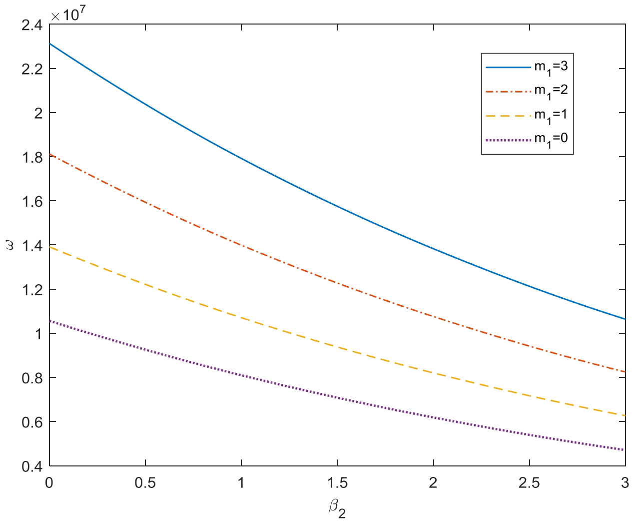

Fig. 17 Torsional frequency FG nano-cone versus β_2 length for $\beta_2 = m_2 = 0$ and $l = \mu = 1\,nm$

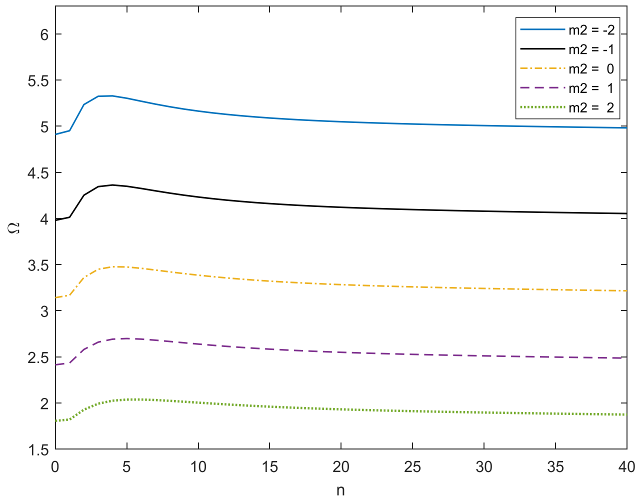

Fig. 15 Torsional frequency FG nano-cone versus n length for $\beta_1 = \beta_2 = m_1 = 0$ and $l = \mu = 1\,nm$

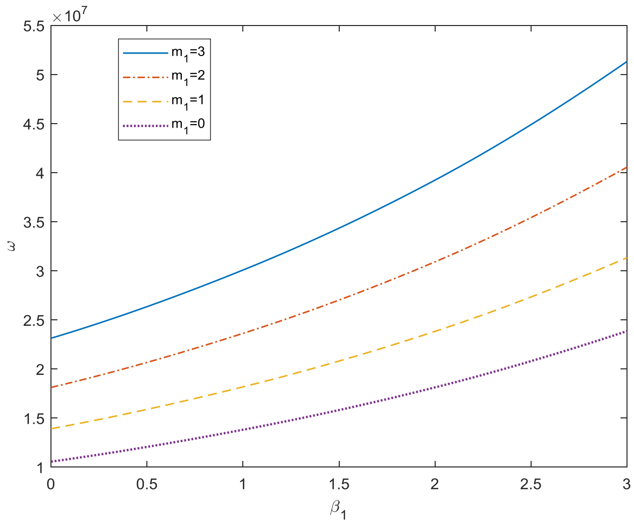

Fig. 16 Torsional frequency FG nano-cone versus β_1 length for $\beta_2 = m_2 = 0$ and $l = \mu = 1\,nm$

5. Conclusions

In this paper, based on nonlocal strain gradient theory and Hamilton's principle, the Navier equations of bi-directional FG nano-cone with arbitrary section subjected torsional loading were derived. The effect of different parameters such as the nonlocal parameter, the cross-section change parameter, FG parameters were considered. In this paper, we use an arbitrary function for all of the material parameters. For numerical results, in the final section we use a function for material properties that this function related to β_1, β_2, m_1 and m_2. By increasing β_1 and m_1, the shear modules increase along *z* and *r* directions. β_2 and m_2 have a straight relationship with density. Comparison of the results obtained from this paper with those obtained from literature proves the efficiency and accuracy of this paper.

It was found that:

- β_1 and m_1 have a straight relation with the torsional frequency of nano-cone.
- β_2 and m_1 have an inverse relation with the torsional frequency of nano-cone.
- By increasing the length of nano-cone, the nonlocal elasticity theory, the strain gradient theory and classical elasticity theory approach to each other's.
- Results show the small-scale effect vanishes if the length of the nano-cone exceeds 50 nm.
- The nonlocal and strain gradient theories, in comparison to the classical elasticity theory, predict softer and stiffer behavior for nano-structures.
- The results indicate that small n has a significant effect on the natural torsional frequency of bidirectional FG nano-cone. In large values of n, there is no effect on the torsional frequency.

References

Abazari, A.M., Safavi, S.M., Rezazadeh, G. and Villanueva, L.G. (2015), "Size Effects on Mechanical Properties of Micro/Nano Structures", *arXiv preprint arXiv:*150801322.

Adeli, M.M., Hadi, A., Hosseini, M. and Gorgani, H.H. (2017), "Torsional vibration of nano-cone based on nonlocal strain gradient elasticity theory", *Eur. Phys. J. Plus*, **132**(9), 393. https://doi.org/10.1140/epjp/i2017-11688-0

Akbas, S.D. (2018), "Bending of a cracked functionally graded nanobeam", *Adv. Nano Res*., *Int. J.,* **6**(3), 219-242. https://doi.org/10.12989/anr.2018.6.3.219

Asemi, S.R. and Farajpour, A. (2014), "Vibration characteristics of double-piezoelectric-nanoplate-systems", *Micro. Nano Lett.*, **9**(4), 280-285. https://doi.org/10.1049/mnl.2013.0741

Aydogdu, M., Arda, M. and Filiz, S. (2018), "Vibration of axially functionally graded nano rods and beams with a variable nonlocal parameter", *Adv. Nano Res*., *Int. J.,* **6**(3), 257-278. https://doi.org/10.12989/anr.2018.6.3.257

Barati, M.R. (2017), "Vibration analysis of FG nanoplates with nanovoids on viscoelastic substrate under hygro-thermo-mechanical loading using nonlocal strain gradient theory", *Struct. Eng. Mech., Int. J.*, **64**(6), 683-693. https://doi.org/10.12989/sem.2017.64.6.683

Bodaghi, M., Noroozi, R., Zolfagharian, A., Fotouhi, M. and Norouzi, S. (2019), "4D printing self-morphing structures", *Materials*, **2**(8), 1353. https://doi.org/10.3390/ma12081353

Demir, C. and Civalek, Ö. (2013), "Torsional and longitudinal frequency and wave response of microtubules based on the nonlocal continuum and nonlocal discrete models", *Appl. Math. Modell*., **37**(22), 9355-9367. https://doi.org/10.1016/j.apm.2013.04.050

Dowling, A.P.J.M.T. (2004), "Development of nanotechnologies", *Mater. Today,* **7**(12), 30-35. https://doi.org/10.1016/S1369-7021(04)00628-5

Ebrahimi, F. and Barati, M.R. (2017a), "A nonlocal strain gradient refined beam model for buckling analysis of size-dependent shear-deformable curved FG nanobeams", *Compos. Struct.,* **159**, 174-182. https://doi.org/10.1016/j.compstruct.2016.09.058

Ebrahimi, F. and Barati, M.R. (2017b), "Vibration analysis of viscoelastic inhomogeneous nanobeams incorporating surface and thermal effects", *Appl. Phys. A*, **123**(1), 5. https://doi.org/10.1007/s00339-016-0511-z

Ebrahimi, F. and Barati, M.R. (2018a), "Buckling analysis of nonlocal strain gradient axially functionally graded nanobeams resting on variable elastic medium", *Proc. Inst. Mech. Eng. Pt. C J. Mechan. Eng. Sci.* **232**(11), 2067-2078. https://doi.org/10.1177/0954406217713518

Ebrahimi, F. and Barati, M.R. (2018b), "Stability analysis of functionally graded heterogeneous piezoelectric nanobeams based on nonlocal elasticity theory", *Adv. Nano Res*., *Int. J.,* **6**(2), 93-112. https://doi.org/10.12989/anr.2018.6.2.093

Ebrahimi, F. and Dabbagh, A. (2017), "Nonlocal strain gradient based wave dispersion behavior of smart rotating magneto-electro-elastic nanoplates", *Mater. Res. Express*, **4**(2), 025003. https://doi.org/10.1088/2053-1591/aa55b5

Ebrahimi, F. and Fardshad, R.E. (2018), "Modeling the size effect on vibration characteristics of functionally graded piezoelectric nanobeams based on Reddy's shear deformation beam theory", *Adv. Nano Res*., *Int. J.,* **6**(2), 113-133. https://doi.org/10.12989/anr.2018.6.2.113

Ebrahimi, F. and Haghi, P. (2017), "Wave propagation analysis of rotating thermoelastically-actuated nanobeams based on nonlocal strain gradient theory", *Acta Mech. Solida Sin*.. **30**(6), 647-657. https://doi.org/10.1016/j.camss.2017.09.007

Ebrahimi, F. and Haghi, P. (2018), "Elastic wave dispersion modelling within rotating functionally graded nanobeams in thermal environment", *Adv. Nano Res*., *Int. J.,* **6**(3), 201-217. https://doi.org/10.12989/anr.2018.6.3.201

Ebrahimi, F. and Salari, E. (2018), "Effect of non-uniform temperature distributions on nonlocal vibration and buckling of inhomogeneous size-dependent beams", *Adv. Nano Res*., *Int. J.,* **6**(4), 377-397. https://doi.org/10.12989/anr.2018.6.4.377

Ebrahimi, F., Barati, M.R. and Haghi, P. (2017a), "Thermal effects on wave propagation characteristics of rotating strain gradient temperature-dependent functionally graded nanoscale beams", *J. Therm. Stresses*, **40**(5), 535-547. https://doi.org/10.1080/01495739.2016.1230483

Ebrahimi, F., Daman, M. and Jafari, A. (2017b), "Nonlocal strain gradient-based vibration analysis of embedded curved porous piezoelectric nano-beams in thermal environment", *Smart Struct. Syst*., *Int. J.,* **20**(6), 709-728. https://doi.org/10.12989/sss.2017.20.6.709

El-Borgi, S., Rajendran, P., Friswell, M., Trabelssi, M. and Reddy, J. (2018), "Torsional vibration of size-dependent viscoelastic rods using nonlocal strain and velocity gradient theory", *Compos. Struct.*, **186**, 274-292. https://doi.org/10.1016/j.compstruct.2017.12.002

Eltaher, M., Fouda, N., El-midany, T., Sadoun, A.J.J.o.t.B.S.o.M.S. and Engineering (2018), "Modified porosity model in analysis of functionally graded porous nanobeams", *J. Brazil. Soc. Mech. Sci. Eng.,* **40**(3), 141. https://doi.org/10.1007/s40430-018-1065-0

Eringen, A.C. (1972), "Nonlocal polar elastic continua", *Int. J. Eng. Sci.,* **10**(1), 1-16. https://doi.org/10.1016/0020-7225(72)90070-5

Eringen, A.C. (1983), "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves", *J. Appl. Phys.*, **54**(9), 4703-4710. https://doi.org/10.1063/1.332803

Eringen, A.C. (2002), *Nonlocal continuum field theories*, Springer Science & Business Media

Farajpour, A. and Rastgoo, A. (2017), "Influence of carbon nanotubes on the buckling of microtubule bundles in viscoelastic cytoplasm using nonlocal strain gradient theory", *Results Phys.*, **7**, 1367-1375. https://doi.org/10.1016/j.rinp.2017.03.038

Farajpour, M.R., Rastgoo, A., Farajpour, A. and Mohammadi, M. (2016), "Vibration of piezoelectric nanofilm-based electromechanical sensors via higher-order non-local strain gradient theory", *Micro. Nano Lett.*, **11**(6), 302-307. https://doi.org/10.1049/mnl.2016.0081

Farajpour, A., Rastgoo, A. and Farajpour, M. (2017a), "Nonlinear buckling analysis of magneto-electro-elastic CNT-MT hybrid nanoshells based on the nonlocal continuum mechanics", *Compos. Struct*., **180**, 179-191. https://doi.org/10.1016/j.compstruct.2017.07.100

Farajpour, A., Rastgoo, A. and Mohammadi, M. (2017b), "Vibration, buckling and smart control of microtubules using piezoelectric nanoshells under electric voltage in thermal environment", *Physica B: Condensed Matter*, **509**, 100-114. https://doi.org/10.1016/j.physb.2017.01.006

Farajpour, M., Shahidi, A., Hadi, A. and Farajpour, A. (2018a), "Influence of initial edge displacement on the nonlinear vibration, electrical and magnetic instabilities of magneto-electro-elastic nanofilms", *Mech. Adv. Mater. Struct.*, **26**(17), 1469-1481. https://doi.org/10.1080/15376494.2018.1432820

Farajpour, M., Shahidi, A., Tabataba'i-Nasab, F. and Farajpour, A. (2018b), "Vibration of initially stressed carbon nanotubes under magneto-thermal environment for nanoparticle delivery via higher-order nonlocal strain gradient theory", *Eur Phys J Plus*. **133**(6), 219. https://doi.org/10.1140/epjp/i2018-12039-5

Farajpour, M.R., Shahidi, A. and Farajpour, A. (2018c), "Resonant frequency tuning of nanobeams by piezoelectric nanowires under thermo-electro-magnetic field: a theoretical study", *Micro. Nano Lett.*, **13**(11), 1627-1632. https://doi.org/10.1049/mnl.2018.5286

Goyal, A. and Soni, P.J.M.L. (2018), "Functionally graded nanocrystalline silicon powders by mechanical alloying", **214**, 111-114. https://doi.org/10.1016/j.matlet.2017.11.114

Guo, S., He, Y., Liu, D., Lei, J., Shen, L. and Li, Z. (2016), "Torsional vibration of carbon nanotube with axial velocity and velocity gradient effect", *Int. J. Mech. Sci.*, **119**, 88-96. https://doi.org/10.1016/j.ijmecsci.2016.09.036

Hadi, A., Nejad, M.Z. and Hosseini, M. (2018a), "Vibrations of three-dimensionally graded nanobeams", *Int. J. Mech. Sci.*, **128**, 12-23. https://doi.org/10.1016/j.ijengsci.2018.03.004

Hadi, A., Rastgoo, A., Haghighipour, N. and Bolhassani, A. (2018b), "Numerical modelling of a spheroid living cell membrane under hydrostatic pressure", *J. Stat. Mech. Theory Experiment.*, **2018**(8), 083501. https://doi.org/10.1088/1742-5468/aad369

Hosseini, M., Shishesaz, M., Tahan, K.N. and Hadi, A. (2016), "Stress analysis of rotating nano-disks of variable thickness made of functionally graded materials", *Int. J. Eng. Sci.*, **109**, 29-53. https://doi.org/10.1016/j.ijengsci.2016.09.002

Hosseini, M., Gorgani, H.H., Shishesaz, M. and Hadi, A. (2017), "Size-dependent stress analysis of single-wall carbon nanotube based on strain gradient theory", *Int. J. Appl. Mech.*, **9**(6), 1750087. https://doi.org/10.1142/S1758825117500879

Hosseini, M., Hadi, A., Malekshahi, A. and Shishesaz, M. (2018), "A review of size-dependent elasticity for nanostructures", *J. Computat. Appl. Mech.*, **49**(1), 197-211. https://doi.org/10.22059/JCAMECH.2018.259334.289

Hosseini, M., Shishesaz, M. and Hadi, A. (2019), "Thermoelastic analysis of rotating functionally graded micro/nanodisks of variable thickness", *Thin-Wall. Struct*., **134**, 508-523. https://doi.org/10.1016/j.tws.2018.10.030

Kafadar, C. and Eringen, A.C. (1971a), "Micropolar media—I the classical theory", *Int. J. Eng. Sci*., **9**(3), 271-305. https://doi.org/10.1016/0020-7225(71)90040-1

Kafadar, C. and Eringen, A.C. (1971b), "Micropolar media—II the relativistic theory", *Int. J. Eng. Sci*., **9**(3), 307-329. https://doi.org/10.1016/0020-7225(71)90041-3

Karami, B., Janghorban, M. and Tounsi, A. (2018a), "Nonlocal strain gradient 3D elasticity theory for anisotropic spherical nanoparticles", *Steel Compos. Struct., Int. J.*, **27**(2), 201-216. https://doi.org/10.12989/scs.2018.27.2.201

Karami, B., Shahsavari, D. and Janghorban, M. (2018b), "Wave propagation analysis in functionally graded (FG) nanoplates under in-plane magnetic field based on nonlocal strain gradient theory and four variable refined plate theory", *Mech. Adv. Mater. Struct*., **25**(12), 1047-1057. https://doi.org/10.1080/15376494.2017.1323143

Karami, B., Shahsavari, D., Janghorban, M. and Li, L. (2018c), "Wave dispersion of mounted graphene with initial stress", *Thin-Wall. Struct*., **122**, 102-111. https://doi.org/10.1016/j.tws.2017.10.004

Karami, B., Shahsavari, D. and Li, L. (2018d), "Hygrothermal wave propagation in viscoelastic graphene under in-plane magnetic field based on nonlocal strain gradient theory", *Physica E*, **97**, 317-327. https://doi.org/10.1016/j.physe.2017.11.020

Li, C. (2014), "Torsional vibration of carbon nanotubes: Comparison of two nonlocal models and a semi-continuum model", *Int. J. Mech. Sci*., **82**, 25-31. https://doi.org/10.1016/j.ijmecsci.2014.02.023

Li, L. and Hu, Y. (2016), "Nonlinear bending and free vibration analyses of nonlocal strain gradient beams made of functionally graded material", *Int. J. Eng. Sci*., **107**, 77-97. https://doi.org/10.1016/j.ijengsci.2016.07.011

Li, L. and Hu, Y. (2017), "Torsional vibration of bi-directional functionally graded nanotubes based on nonlocal elasticity theory", *Compos. Struct*., **172**, 242-250. https://doi.org/10.1016/j.compstruct.2017.03.097

Li, L., Hu, Y. and Ling, L. (2015), "Flexural wave propagation in small-scaled functionally graded beams via a nonlocal strain gradient theory", *Compos. Struct*., **133**, 1079-1092. https://doi.org/10.1016/j.compstruct.2015.08.014

Li, L., Li, X. and Hu, Y. (2016), "Free vibration analysis of nonlocal strain gradient beams made of functionally graded material", *Int. J. Eng. Sci*., **102**, 77-92. https://doi.org/10.1016/j.ijengsci.2016.02.010

Li, L., Tang, H. and Hu, Y. (2018), "Size-dependent nonlinear vibration of beam-type porous materials with an initial geometrical curvature", *Compos. Struct*., **184**, 1177-1188. https://doi.org/10.1016/j.compstruct.2017.10.052

Lim, C., Zhang, G. and Reddy, J. (2015), "A higher-order nonlocal elasticity and strain gradient theory and its applications in wave propagation", *J. Mech. Phys. Solids*, **78**, 298-313. https://doi.org/10.1016/j.jmps.2015.02.001

Lu, L., Guo, X. and Zhao, J. (2017a), "Size-dependent vibration analysis of nanobeams based on the nonlocal strain gradient theory", *Int. J. Eng. Sci*., **116**, 12-24. https://doi.org/10.1016/j.ijengsci.2017.03.006

Lu, L., Guo, X. and Zhao, J. (2017b), "A unified nonlocal strain gradient model for nanobeams and the importance of higher order terms", *Int. J. Eng. Sci*., **119**, 265-277. https://doi.org/10.1016/j.ijengsci.2017.06.024

Mehralian, F., Beni, Y.T. and Zeverdejani, M.K. (2017a), "Calibration of nonlocal strain gradient shell model for buckling analysis of nanotubes using molecular dynamics simulations", *Physica B, Condens. Matter*, **521**, 102-111. https://doi.org/10.1016/j.physb.2017.06.058

Mehralian, F., Beni, Y.T. and Zeverdejani, M.K. (2017b), "Nonlocal strain gradient theory calibration using molecular dynamics simulation based on small scale vibration of nanotubes", *Physica B, Condens. Matter*, **514**, 61-69. https://doi.org/10.1016/j.physb.2017.03.030

Mindlin, R. and Eshel, N. (1968), "On first strain-gradient theories in linear elasticity", *Int. J. Solids Struct*., **4**(1), 109-124. https://doi.org/10.1016/0020-7683(68)90036-X

Miyamoto, Y., Kaysser, W., Rabin, B., Kawasaki, A. and Ford, R.G. (2013), *Functionally graded materials: design, processing and applications*, Springer Science & Business Media

Nejad, M.Z. and Hadi, A. (2016a), "Eringen's non-local elasticity theory for bending analysis of bi-directional functionally graded Euler–Bernoulli nano-beams", *Int. J. Eng. Sci*., **106**, 1-9. https://doi.org/10.1016/j.ijengsci.2016.05.005

Nejad, M.Z. and Hadi, A. (2016b), "Non-local analysis of free vibration of bi-directional functionally graded Euler–Bernoulli nano-beams", *Int. J. Eng. Sci*., **105**, 1-11. https://doi.org/10.1016/j.ijengsci.2016.04.011

Nejad, M.Z., Hadi, A. and Rastgoo, A. (2016), "Buckling analysis of arbitrary two-directional functionally graded Euler–Bernoulli nano-beams based on nonlocal elasticity theory", *Int. J. Eng. Sci*., **103**, 1-10. https://doi.org/10.1016/j.ijengsci.2016.03.001

Nejad, M.Z., Alamzadeh, N. and Hadi, A. (2018), "Thermoelastoplastic analysis of FGM rotating thick cylindrical pressure vessels in linear elastic-fully plastic condition", *Compos. Part B-Eng*., **154**, 410-422. https://doi.org/10.1016/j.compositesb.2018.09.022

Noroozi, R. and Ataee, A. (2018), "Behavioral Optimization of Pseudo-Neutral Hole in Hyperelastic Membranes Using Functionally graded Cables", *J. Computat. Appl. Mech.*, **49**(2), 282-291. https://doi.org/10.22059/JCAMECH.2018.268899.338

Petit, C., Montanaro, L. and Palmero, O. (2018), "Functionally graded ceramics for biomedical application: Concept, manufacturing, and properties", *Int. J. Appl. Ceram. Technol.*, **15**(4), 820-840. https://doi.org/10.1111/ijac.12878

Phung-Van, P., Ferreira, A., Nguyen-Xuan, H. and Wahab, M.A. (2017a), "An isogeometric approach for size-dependent geometrically nonlinear transient analysis of functionally graded nanoplates", *Compos. Part B-Eng*., **118**, 125-134. https://doi.org/10.1016/j.compositesb.2017.03.012

Phung-Van, P., Lieu, Q.X., Nguyen-Xuan, H. and Wahab, M.A. (2017b), "Size-dependent isogeometric analysis of functionally graded carbon nanotube-reinforced composite nanoplates", *Compos. Struct*., **166**, 120-135. https://doi.org/10.1016/j.compstruct.2017.01.049

Phung-Van, P., Tran, L.V., Ferreira, A., Nguyen-Xuan, H. and Abdel-Wahab, M.J.N.D. (2017c), "Nonlinear transient isogeometric analysis of smart piezoelectric functionally graded material plates based on generalized shear deformation theory under thermo-electro-mechanical loads", *Nonliear Dyn.,* **87**(2), 879-894. https://doi.org/10.1007/s11071-016-3085-6

Phung-Van, P., Thai, C.H., Nguyen-Xuan, H. and Wahab, M.A. (2019), "Porosity-dependent nonlinear transient responses of functionally graded nanoplates using isogeometric analysis", *Compos. Part B-Eng*., **164**, 215-225. https://doi.org/10.1016/j.compositesb.2018.11.036

Pompe, W., Worch, H., Epple, M., Friess, W., Gelinsky, M., Greil, P., Hempel, U., Scharnweber, D., Schulte, K.J.M.S. and A, E. (2003), "Functionally graded materials for biomedical applications", *Mater. Sci. Eng. A*, **362**(1-2), 40-60. https://doi.org/10.1016/S0921-5093(03)00580-X

Poole Jr, C.P. and Owens, F.J. (2003), *Introduction to nanotechnology*, John Wiley & Sons

Rahmani, O., Norouzi, S., Golmohammadi, H. and Hosseini, S. (2017), "Dynamic response of a double, single-walled carbon nanotube under a moving nanoparticle based on modified nonlocal elasticity theory considering surface effects", *Mech. Adv. Mater. Struct.,* **24**(15), 1274-1291. https://doi.org/10.1080/15376494.2016.1227504

Sahmani, S. and Aghdam, M. (2017), "A nonlocal strain gradient hyperbolic shear deformable shell model for radial postbuckling analysis of functionally graded multilayer GPLRC nanoshells", *Compos. Struct*., **178**, 97-109. https://doi.org/10.1016/j.compstruct.2017.06.062

She, G.-L., Yuan, F.-G., Ren, Y.-R., Liu, H.-B. and Xiao, W.-S. (2018), "Nonlinear bending and vibration analysis of functionally graded porous tubes via a nonlocal strain gradient theory", *Compos. Struct*., **203**, 614-623. https://doi.org/10.1016/j.compstruct.2018.07.063

Shen, Y., Chen, Y. and Li, L. (2016), "Torsion of a functionally graded material", *Int. J. Eng. Sci*., **109**, 14-28. https://doi.org/10.1016/j.ijengsci.2016.09.003

Shishesaz, M., Hosseini, M., Tahan, K.N. and Hadi, A. (2017), "Analysis of functionally graded nanodisks under thermoelastic loading based on the strain gradient theory", *Acta Mechanica*, **228**(12), 4141-4168. https://doi.org/10.1007/s00707-017-1939-8

Shu, C. and Chew, Y. (1998), "On the equivalence of generalized differential quadrature and highest order finite difference scheme", *Comput. Method Appl. M*, **155**(3-4), 249-260. https://doi.org/10.1016/S0045-7825(97)00150-3

Simsek, M. (2016), "Nonlinear free vibration of a functionally graded nanobeam using nonlocal strain gradient theory and a novel Hamiltonian approach", *Int. J. Eng. Sci*., **105**, 12-27. https://doi.org/10.1016/j.ijengsci.2016.04.013

Thai, C.H., Ferreira, A., Wahab, M.A. and Nguyen-Xuan, H. (2018), "A moving Kriging meshfree method with naturally stabilized nodal integration for analysis of functionally graded material sandwich plates", *Acta Mech*., **229**(7), 2997-3023. https://doi.org/10.1007/s00707-018-2156-9

Thanh, C.-L., Phung-Van, P., Thai, C.H., Nguyen-Xuan, H. and Wahab, M.A. (2018), "Isogeometric analysis of functionally graded carbon nanotube reinforced composite nanoplates using modified couple stress theory", *Compos. Struct*., **184**, 633-649. https://doi.org/10.1016/j.compstruct.2017.10.025

Thanh, C.-L., Ferreira, A. and Wahab, M.A. (2019a), "A refined size-dependent couple stress theory for laminated composite micro-plates using isogeometric analysis", *Thin-Wall. Struct*., **145**, 106427. https://doi.org/10.1016/j.tws.2019.106427

Thanh, C.-L., Tran, L.V., Bui, T.Q., Nguyen, H.X. and Abdel-Wahab, M. (2019b), "Isogeometric analysis for size-dependent nonlinear thermal stability of porous FG microplates", *Compos Struct*., **221**, 110838. https://doi.org/10.1016/j.compstruct.2019.04.010

Thanh, C.-L., Tran, L.V., Vu-Huu, T. and Abdel-Wahab, M. (2019c), "The size-dependent thermal bending and buckling analyses of composite laminate microplate based on new modified couple stress theory and isogeometric analysis", *Comput. Method Appl. M*, **350**, 337-361. https://doi.org/10.1016/j.cma.2019.02.028

Thanh, C.-L., Tran, L.V., Vu-Huu, T., Nguyen-Xuan, H. and Abdel-Wahab, M. (2019d), "Size-dependent nonlinear analysis and damping responses of FG-CNTRC micro-plates", *Comput. Method Appl. M*, **353**, 253-276. https://doi.org/10.1016/j.cma.2019.05.002

Toupin, R.A. (1962), "Elastic materials with couple-stresses", *Arch. Ration. Mech. Anal*., **11**(1), 385-414.

Xu, X.-J., Zheng, M.-L. and Wang, X.-C. (2017), "On vibrations of nonlocal rods: Boundary conditions, exact solutions and their asymptotics", *Int. J. Eng. Sci*., **119**, 217-231. https://doi.org/10.1016/j.ijengsci.2017.06.025

Yayli, M.Ö. (2018), "Torsional vibrations of restrained nanotubes using modified couple stress theory", *Microsyst. Technol*., **24**(8), 3425-3435. https://doi.org/10.1007/s00542-018-3735-3

Zamani Nejad, M., Jabbari, M. and Hadi, A. (2017), "A review of functionally graded thick cylindrical and conical shells", *J. Computat. Appl. Mech.,* **48**(2), 357-370. https://doi.org/10.22059/JCAMECH.2017.247963.220

Zhu, X. and Li, L. (2017a), "Closed form solution for a nonlocal strain gradient rod in tension", *Int. J. Eng. Sci*., **119**, 16-28. https://doi.org/10.1016/j.ijengsci.2017.06.019

Zhu, X. and Li, L. (2017b), "On longitudinal dynamics of nanorods", *Int. J. Eng. Sci*., **120**, 129-145. https://doi.org/10.1016/j.ijengsci.2017.08.003

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