Advances in Nano Research, *Vol. 6 No. 4* (*2018*) *323-337*
DOI: https://doi.org/10.12989/anr.2018.6.4.323

Finite strain nonlinear longitudinal vibration of nanorods

Mehmet Eren [1a] and Metin Aydogdu [*2]

[1] *Institute of Science*, *Trakya University*, *22030 Edirne*, *Turkey*
[2] *Department of Mechanical Engineering*, *Trakya University*, *22030 Edirne*, *Turkey*

(*Received May 24*, *2018*, *Revised August 31*, *2018*, *Accepted October 12*, *2018*)

Abstract. The nonlinear free vibration of a nanorod subjected to finite strain is investigated. The governing equation of motion in material configuration in terms of displacement is determined. By means of Galerkin method, the Fourier series solutions satisfying some typical boundary conditions are determined. The amplitude-frequency relationship and interaction between the modes are studied. The effects of nonlocal elasticity are shown for different length of nanotubes and nonlocal parameter. The results show that nonlocal effects lead to additional internal modal interaction for nanorod vibrations.

Keywords: nonlocal elasticity; mode interactions; nonlocal elastic rod model; nonlinear axial vibration; finite strain

1. Introduction

Nanoscience and nanotechnology are the one of the most important topics nowadays. Carbon nanotubes are the main element in many applications at nano-length scale. Single and multi-walled carbon nanotubes are used in various nano-electromechanical systems and material science. Understanding the mechanical behavior of nanotubes is an important subject for efficient design.

Generally molecular dynamic models and continuum models have been used in the modeling of carbon nanotubes. Mechanics of nanoscale structures depend on their size. This size dependence is lost with increasing dimensions. Different nonlocal elasticity models were considered to include the size dependent behavior of nanostructures.

Eringen (1976, 1983) proposed a stress type nonlocal elasticity model. In his model it is assumed that the stress at a reference point is a function of the strain field at every point of the continuum. This model considers the interactions of the nearest atoms in the elastic body. Eringen (1983) utilized the lattice dynamic results in one dimensional wave propagation at the end of the first Broullin zone in order to validate his stress gradient nonlocal theory. Recently, bending, buckling and wave propagation in nano structures have been studied using the nonlocal elasticity theory (Peddieson *et al*. 2003, Sudak 2003, Wang and Hu 2005, Aydogdu 2009, Chaht *et al*. 2015, Rahmani 2017, Ebrahimi and Mahmoodi 2018). It is shown that the nonlocal theory results for statics and dynamics of nanotubes are in good agreement with molecular dynamic results.

∗Corresponding author, Professor, E-mail: metina@trakya.edu.tr
[a] M.Sc., E-mail: mehmeteren17@gmail.com

http://www.techno-press.org/?journal=anr&subpage=5 ISSN: 2287-237X (Print), 2287-2388 (Online)

Linear axial vibration of carbon nanotubes was studied using the nonlocal stress gradient continuum models by Aydogdu (2009, 2014), Aydogdu and Elishakoff (2014) and Danesh *et al*. (2012), Aydogdu and Arda (2016). Adhikari *et al*. (2014) investigated the frequency domain analysis of nonlocal rods embedded in an elastic medium. Axial vibration of embedded nanorods under transverse magnetic field effects via nonlocal elastic continuum theory was studied by Murmu *et al*. (2014). It is shown in previous studies that the deformation of CNTs is nonlinear in nature. Fu *et al*. (2006) studied the nonlinear free vibration of embedded multiwalled CNTs. They showed that the nonlinear natural frequency increases with increasing vibration amplitude. Nonlinear free vibration of embedded double-walled carbon nanotubes based on nonlocal Timoshenko beam theory was investigated by Ke *et al*. (2009). Yang *et al*. (2010) investigated the nonlinear free vibration of single-walled carbon nanotubes using Timoshenko beam theory. Applicability of FEM in studying the free nonlinear vibration of CNTs is investigated by Ansari *et al*. (2011). Cigeroglu and Samandari (2014) investigated the nonlinear free vibration of curved double-walled carbon nanotubes using the differential quadrature method. Nonlinear free vibration of double walled carbon nanotubes by using describing function method with multiple trial functions was studied by Cigeroglu and Samandari (2012). Ghayesh (2014) investigated nonlinear size-dependent behavior of single-walled carbon nanotubes. Axial nonlinear vibration of rods is studied by Mousavi and Fariborz (2012) using local elasticity theory. Nonlinear vibration of a zigzag nanotube embedded in a polymer matrix has been studied by Besseghier *et al*. (2015). Nonlinear free vibration of double-walled carbon nanotubes was studied by using Euler-Bernoulli beam theory and multiple scales perturbation method (Hajnayeb and Khadem 2015). A nonlocal sinusoidal shear deformation theory has been used for the nonlinear vibration of single walled carbon nanotubes using Timoshenko beam theory and differential quadrature method (Pour *et al*. 2015). Nonlinear transverse vibration of tensioned nanobeams has been investigated by using nonlocal beam theory (Bagdatli 2015). Fernandes *et al*. (2017) studied nonlinear vibration of embedded nanotubes. Another size dependent model called couple stress theory used in order to investigate nonlinear dynamics of micro beams (Ghayesh *et al*. 2013a, b, c, 2014, 2016, Farokhi *et al*. 2013, Faroki and Ghayesh 2015, Ghayesh and Farokhi 2015a), micro plates (Gholipour *et al*. 2015, Farokhi and Ghayesh 2015, Ghayesh and Farokhi 2015b) and functionally graded micro structures (Ghayesh 2018a, b, c). Nonlinear forced vibration of micro beams studied using strain gradient theory (Ghayesh *et al*. 2013d). According to authors best knowledge, axial nonlinear vibration of nanotubes has rarely been considered in the previous studies.

In this paper, nonlinear finite strain longitudinal vibration of the nanorods is investigated. The amplitude-frequency relationship, also called backbone curves for three different boundary conditions are presented. Galerkin method is utilized in the formulation of the problem. The nonlocal elasticity theory is utilized in the modeling. The difference between classical elasticity and the nonlocal elasticity is discussed in detail.

2. Analysis

2.1 Equations of motion of nanorods using nonlocal elasticity

Consider a nanorod of length L and diameter d. The deformation gradient tensor can be written as (Lai *et al*. 2010)

$$F = \nabla U + I \tag{1}$$

where U and I is the displacement vector and the unit matrix, respectively. It should be noted that vector and tensor quantities are denoted in bold letters in the present study. The equations of motion of the nanorod can be written as (Lai *et al*. 2010)

$$\nabla \cdot \left[\boldsymbol{S}\boldsymbol{F}^{T}\right] = \rho_0 \frac{\partial^2 \boldsymbol{U}}{\partial t^2} \tag{2}$$

where ρ_0 is the density, t is the time and $\boldsymbol{S}$ is the second Piola-Kirchhoff stress tensor. Hooke's law can be written as (Lai *et al*. 2010)

$$\boldsymbol{S} = \boldsymbol{c}\boldsymbol{E} \tag{3}$$

where $\boldsymbol{c}$ is the fourth order elasticity tensor and $\boldsymbol{E}$ is the Green strain tensor defined as (Lai *et al*. 2010)

$$\boldsymbol{E} = \frac{1}{2}\left[\boldsymbol{F}^{T}\boldsymbol{F} - \boldsymbol{I}\right] \tag{4}$$

If the axial displacement $U(x,t)$ is the only nonzero deformation, then the gradient deformation tensor can be written as

$$\boldsymbol{F} = \begin{bmatrix} 1+\partial \mathrm{U}/\partial \mathrm{x} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

From Eq. (4) axial strain can be written as

$$E_{xx} = \left(1 + \frac{1}{2}\frac{\partial U}{\partial x}\right)\frac{\partial U}{\partial x} \tag{6}$$

The stress-strain relationships (3) becomes

$$S_{ij} = \frac{Y}{1+\nu}\left[E_{ij} + \frac{\nu}{1-2\nu}E_{kk}\delta_{ij}\right] \tag{7}$$

where Y and ν are the elasticity modulus and Poisson's ratio, respectively and δ_{ij} is the Kronecker delta. Substituting Eq. (6) into Eq. (7), following stress components are determined

$$S_{xx} = \frac{Y(1-\nu)}{(1+\nu)(1-2\nu)}\left(1 + \frac{1}{2}\frac{\partial U}{\partial x}\right)\frac{\partial U}{\partial x} \tag{8}$$

$$S_{yy} = S_{zz} = \frac{Y\nu}{(1+\nu)(1-2\nu)}\left(1 + \frac{1}{2}\frac{\partial U}{\partial x}\right)\frac{\partial U}{\partial x} \tag{9}$$

The stress tensor in the nonlocal elasticity is function of strain at all points of the body and defined as (Eringen 1983)

$$S_{ij}(\mathbf{x}) = \int_V \chi(|x - x'|, \gamma) T_{ij} dV(x'), \quad \forall x \in V, \tag{10}$$

where S_{ij} are the components of nonlocal stress tensor, T_{ij} are the classical macroscopic stress tensor components at point x. $\chi(|x - x'|, \gamma)$ is the nonlocal modulus or attenuation function related to the long range interactions of the points in the continuum and $|x - x'|$ is the Euclidean distance. $\gamma = e_0 a / l$, where a is an internal characteristic length of the nanorod, l is an external characteristic length (wave length, crack length or dimensions of member) and e_0 is a constant. As generally assumed in the literature, in this study, $0 \le (e_0 a)^2 \le 2$ is used. In the remaining part of this study, $\mu = (e_0 a)^2$ will be used. The nonlocal constitutive relations can be given as (Eringen 1983)

$$(1 - \mu \nabla^2) S_{kl} = \lambda_c E_{rr} \delta_{kl} + 2\mu_c E_{kl} \tag{11}$$

where λ_c and μ_c are the Lame constants. One dimensional form of Eq. (11) can be written as

$$(1 - \mu \frac{\partial^2}{\partial x^2}) S_{xx} = Y E_{xx} \tag{12}$$

Integrating Eq. (12) with respect to cross sectional area A of the nanorod gives axial force relation

$$N_x - \mu \frac{\partial^2 N_x}{\partial x^2} = N_x^L \tag{13}$$

where $N_x = \int_A S_{xx} dA$ and N_x^L denote axial force per unit length for the nonlocal elasticity and local elasticity, respectively. Using Eqs. (1), (8), (9) and (13) the following nonlinear equation of motion for the free longitudinal vibration of nanorods can be obtained

$$\left[\left(\frac{\partial \overline{U}}{\partial \zeta}\right)^2 + 2\left(\frac{\partial \overline{U}}{\partial \zeta}\right) + \frac{2}{3}\right] \frac{\partial^2 \overline{U}}{\partial \zeta^2} = \frac{2}{3} \delta \frac{\partial^2 \overline{U}}{\partial t^2} - \frac{2}{3} \delta \mu \frac{\partial^4 \overline{U}}{\partial \zeta^2 \partial t^2} \tag{14}$$

the non-dimensional variables in Eq.(14) are defined as

$$\overline{U} = \frac{U}{L}, \quad \zeta = \frac{x}{L}, \quad \delta = \frac{\rho_0 (1 + \nu)(1 - 2\nu)}{\mu_c (1 - \nu)} \tag{15}$$

the equation of motion in classical elasticity can be obtained by inserting $\mu = 0$ in Eq. (14). Local form of Eq. (14) has been obtained by (Mousavi and Fariborz 2012).

The nonlinear equation of motion (Eq. (14)) may be solved using a numerical method. In the present study, Galerkin method is used. Three different boundary conditions are considered namely: clamped-clamped, free-free and clamped-free boundary conditions.

2.2 Perturbation analysis

2.2.1 Clamped-clamped nanorods

The clamped boundary conditions are defined as

$$\bar{U}(0,t)=\bar{U}(1,t)=0 \tag{16}$$

In the framework of Galerkin Method, displacement field is chosen in the following Fourier series form

$$\bar{U}(\zeta,t)=\sum_{n=1}^{3}A_n\sin(n\pi\zeta)\cos(\omega t) \tag{17}$$

In the Fourier series first three terms (three modes) are considered in the present analysis. This expansion satisfies the boundary conditions given in Eq. (16). The application of Galerkin method gives

$$\frac{2}{T}\int_0^T\int_0^1 R(\zeta,t)\sin(n\pi\zeta)\cos(\omega t)\partial\zeta\partial t=0 \tag{18}$$

where n = 1, 2, 3 and $T = 2\pi/\omega$ where ω is the angular frequency. Substituting Eq. (17) into Eq. (14) leads to following residue $R(\zeta, t)$

$$\begin{aligned}R(\zeta,t)=\ &\pi^2\Big[\left(A_1\pi\cos(\pi\zeta)+2A_2\pi\cos(2\pi\zeta)+3A_3\pi\cos(3\pi\zeta)\right)^2\cos^2(\omega t)\\&+2\left(A_1\pi\cos(\pi\zeta)+2A_2\pi\cos(2\pi\zeta)+3A_3\pi\cos(3\pi\zeta)\right)\cos(\omega t)+\frac{2}{3}\Big]\\&\left[\left(A_1\pi\sin(\pi\zeta)+4A_2\pi\sin(2\pi\zeta)+9A_3\pi\cos(3\pi\zeta)\right)\cos(\omega t)\right.\\&-\frac{2}{3}\lambda^2\left[A_1\pi\sin(\pi\zeta)+A_2\pi\sin(2\pi\zeta)+A_3\pi\cos(3\pi\zeta)\right]\cos(\omega t)\\&-\mu\frac{2}{3}\pi^2\lambda^2\left[A_1\pi\sin(\pi\zeta)+4A_2\pi\sin(2\pi\zeta)+9A_3\pi\cos(3\pi\zeta)\right]\cos(\omega t)\end{aligned} \tag{19}$$

Eq. (18) leads to three algebraic equations

$$\frac{3\pi^4}{4}\left(\frac{1}{8}A_1^3+A_2^2A_1+\frac{9}{4}A_3^2A_1+\frac{3}{8}A_1^2A_3+\frac{3}{2}A_2^2A_3\right)+\frac{1}{3}A_1\left(\pi^2-\lambda^2-\mu\pi^2\lambda^2\right)=0 \tag{20a}$$

$$A_2\left[\frac{3\pi^4}{4}\left(2A_2^2+A_1^2+9A_3^2+3A_1A_3\right)+\frac{1}{3}\left(4\pi^2-\lambda^2-4\mu\pi^2\lambda^2\right)\right]=0 \tag{20b}$$

$$\frac{3\pi^4}{4}\left(\frac{1}{8}A_1^3+\frac{81}{8}A_3^3+\frac{9}{4}A_1^2A_3+9A_2^2A_3+\frac{3}{2}A_1A_2^2\right)+\frac{1}{3}A_3\left(9\pi^2-\lambda^2-9\mu\pi^2\lambda^2\right)=0 \tag{20c}$$

where $\lambda = \omega\sqrt{\delta}$ is the dimensionless frequency parameter. The amplitudes in Eq. (20) are obtained as

$$A_2 = A_3 = 0,\ A_1 = \pm\frac{4\sqrt{2}}{3\pi^2}\sqrt{\lambda^2 - \pi^2 + \mu\pi^2\lambda^2},\ \lambda^2 - \pi^2 + \mu\pi^2\lambda^2 \ge 0, \tag{21a}$$

$$A_1 = A_3 = 0,\ A_2 = \pm\frac{\sqrt{2}}{3\pi^2}\sqrt{\lambda^2 - 4\pi^2 + 4\mu\pi^2\lambda^2},\ \lambda^2 - 4\pi^2 + 4\mu\pi^2\lambda^2 \ge 0, \tag{21b}$$

$$A_1 = A_2 = 0,\ A_3 = \pm\frac{4}{9\sqrt{6}\pi^2}\sqrt{\lambda^2 - 9\pi^2 + 9\mu\pi^2\lambda^2},\ \lambda^2 - 9\pi^2 + 9\mu\pi^2\lambda^2 \ge 0, \tag{21c}$$

It is seen from Eq. (21) that the dimensionless frequency parameter depends on amplitude of the nanorod. The nonlocal effects decrease frequencies and increase the amplitudes. Internal resonance between the first and third mode appears when "A_2 = 0, $A_1 \neq$ 0, $A_3 \neq$ 0" has been substituted into the Eq. (20) which leads to Eq. (21). So that there is an interaction between these modes. Similar results can be found for the first-second and second-third modes. Internal resonance equation for the first and third modes is

$$\frac{\frac{9\pi^2}{16}A_3\left[\frac{1}{2}A_1^2 + 9A_3^2 + \frac{3}{2}A_1A_3\right] + A_3}{1+\mu\pi^2} - \frac{\frac{9\pi^2}{16}A_3\left[\frac{1}{2}A_1^3 + \frac{81}{2}A_3^2 + 9A_1^2A_3\right] + 9A_3}{1+\mu\pi^2} = 0 \tag{22}$$

Frequency equation of internal resonance between the first and the third mode can be written as follows (by setting A_2 = 0 in Eq. (20a))

$$\lambda = \frac{\frac{\pi}{\sqrt{\delta}}\left[\frac{9\pi^2}{16}\left(\frac{1}{2}A_1^2 + 9A_3^2 + \frac{3}{2}A_1A_3\right) + 1\right]^{1/2}}{\left(1+\pi^2\mu\right)^{1/2}} \tag{23}$$

2.2.2 Free-free nanorods

The boundary conditions of the free-free nanorod are $N_x^L = 0$ at x = 0,1. The solution of Eq. (14) is again assumed in the following form

$$\overline{U}(\zeta,t) = \sum_{n=1}^{3} A_n \cos(n\pi\zeta)\cos(\omega t) \tag{24}$$

Again, using Galerkin method gives

$$\frac{3\pi^4}{4}\left(\frac{1}{8}A_1^3 + A_2^2A_1 + \frac{9}{4}A_3^2A_1 - \frac{3}{8}A_1^2A_3 + \frac{3}{2}A_2^2A_3\right) + \frac{1}{3}A_1\left(\pi^2 - \lambda^2 - \mu\pi^2\lambda^2\right) = 0 \tag{25a}$$

$$A_2\left[\frac{3\pi^4}{4}\left(2A_2^2 + A_1^2 + 9A_3^2 + 3A_1A_3\right) + \frac{1}{3}\left(4\pi^2 - \lambda^2 - 4\mu\pi^2\lambda^2\right)\right] = 0 \tag{25b}$$

$$\frac{3\pi^4}{4}\left(-\frac{1}{8}A_1^3+\frac{81}{8}A_3^3+\frac{9}{4}A_1^2A_3+9A_2^2A_3+\frac{3}{2}A_1A_2^2\right)+\frac{1}{3}A_3\left(9\pi^2-\lambda^2-9\mu\pi^2\lambda^2\right)=0 \tag{25c}$$

The solutions of Eqs. (25) and (20) are the same. Therefore, Fig. 1 shows the bifurcation points and backbone curves of free-free boundary conditions. Similar to clamped-clamped boundary conditions equation of internal resonance is written as in the Eq. (25) assuming "$A_2 = 0$, $A_1 \neq 0$, $A_3 \neq 0$".

$$\frac{\frac{9\pi^2}{4}\left[\frac{1}{8}A_1^2A_3+\frac{9}{4}A_3^2-\frac{3}{8}A_1A_3^2\right]+A_3}{1+\mu\pi^2}-\frac{\frac{9\pi^2}{4}A_3\left[-\frac{1}{8}A_1^3+\frac{81}{8}A_3^2+\frac{9}{4}A_1^2A_3\right]+9A_3}{1+9\mu\pi^2}=0 \tag{26}$$

Frequency equation of internal resonance between first and third mode can be written as follows

$$\lambda=\frac{\frac{\pi}{\sqrt{\delta}}\left[\frac{9\pi^2}{16}\left(\frac{1}{2}A_1^2+9A_3^2-\frac{3}{2}A_1A_3\right)+1\right]^{1/2}}{\left(1+\pi^2\mu\right)^{1/2}} \tag{27}$$

It should be noted that since series starts from n=1 in Eq. (24) zero frequencies (rigid body motion) was not observed in this study.

2.2.3 Clamped-free nanorods

The solution for a clamped-free nano rod is taken as

$$\bar{U}(\zeta,t)=\sum_{n=1}^{3}A_n\sin[(2n-1)\pi\zeta/2]\cos(n\pi\zeta)\cos(\omega t) \tag{28}$$

Similar to the clamped-clamped nanorod, the following system of algebraic equations are obtained

$$\begin{aligned}&\frac{3\pi^4}{4}\left(\frac{1}{8}A_1^3+\frac{9}{4}A_2^2A_1+\frac{25}{4}A_3^2A_1+\frac{3}{8}A_1^2A_2+\frac{15}{4}A_1A_2A_3+\frac{45}{8}A_2^2A_3\right)\\&+\frac{1}{3}A_1\left(\frac{\pi^2}{4}-\lambda^2-\mu\frac{\pi^2}{4}\lambda^2\right)=0\end{aligned} \tag{29a}$$

$$\begin{aligned}&\frac{3\pi^4}{64}\left(\frac{1}{8}A_1^3-\frac{9}{8}A_1^2A_3+\frac{45}{4}A_1A_2A_3+\frac{9}{4}A_1^2A_2+\frac{81}{8}A_2^3+\frac{225}{4}A_3^2A_3\right)\\&+\frac{1}{3}A_2\left(\frac{9\pi^2}{4}-\lambda^2-\mu\frac{9\pi^2}{4}\lambda^2\right)=0\end{aligned} \tag{29b}$$

$$\frac{3\pi^4}{4}\left(\frac{45}{8}A_1A_2^2+\frac{15}{8}A_1^2A_2+\frac{25}{4}A_1^2A_3+\frac{225}{4}A_2^2A_3+\frac{625}{8}A_3^3\right)$$
$$+\frac{1}{3}A_3\left(\frac{25}{4}\pi^2-\lambda^2-\mu\frac{25}{4}\pi^2\lambda^2\right)=0 \tag{29c}$$

The nontrivial solutions to the Eq. (29) are

$$A_2=A_3=0,\ A_1=\pm\frac{16\sqrt{2}}{3\pi^2}\sqrt{\lambda^2-\frac{\pi^2}{4}+\mu\frac{\pi^2}{4}\lambda^2},\ \lambda^2-\frac{\pi^2}{4}+\mu\frac{\pi^2}{4}\lambda^2\geq 0, \tag{30a}$$

$$A_1=A_3=0,\ A_2=\pm\frac{16\sqrt{2}}{27\pi^2}\sqrt{\lambda^2-\frac{9\pi^2}{4}+\mu\frac{9\pi^2}{4}\lambda^2},\lambda^2-\frac{9\pi^2}{4}+\mu\frac{9\pi^2}{4}\lambda^2\geq 0, \tag{30b}$$

$$A_1=A_2=0,\ A_3=\pm\frac{16\sqrt{2}}{75\pi^2}\sqrt{\lambda^2-\frac{25\pi^2}{4}+\mu\frac{25\pi^2}{4}\lambda^2},\ \lambda^2-\frac{25\pi^2}{4}+\mu\frac{25\pi^2}{4}\lambda^2\geq 0, \tag{30c}$$

Fig. 2 shows the backbone curves of clamped-free boundary conditions. Bifurcation points can be seen when frequency value equals zero. In the case of clamped-free ends while first-second and second-third modes of vibration show that there is an interaction between them, there is no solution with the real values of first and second modes.

$$\frac{\frac{9\pi^2}{32}\left[\frac{1}{8}A_1^2A_2+\frac{9}{4}A_2^3+\frac{3}{8}A_1A_2^2\right]+\frac{A_2}{2}}{1+\mu\pi^2}-\frac{\frac{9\pi^2}{32}A_3\left[\frac{1}{8}A_1^3+\frac{9}{4}A_1^2A_2+\frac{81}{8}A_2^3\right]+\frac{9A_2}{2}}{1+9\mu\pi^2}=0 \tag{31}$$

The frequency equation for the interaction between first and second modes is written as follows

$$\lambda=\frac{\frac{\pi}{2\sqrt{\delta}}\left[\frac{9\pi^2}{16}\left(\frac{1}{2}A_1^2+9A_2^2+\frac{3}{2}A_1A_2\right)+1\right]^{1/2}}{\left(1+\frac{\pi^2}{4}\mu\right)^{1/2}} \tag{32}$$

Figs. 3-5 show graphical representation of frequency of internal resonances between the different modes of vibration.

3. Numerical results

In this study, backbone curves of axially vibrating nanorods for the first three frequencies are investigated for the boundary conditions studied (Figs. 1-2). In the figures only the fundamental and third frequencies are given due to space limitations. It should be noted that the second mode frequencies change similar to the first and the third mode for the considered parameters. Different nanotube length and nonlocal parameters are considered. The nonlocal parameter and nanotube length has been chosen as $0 \le \mu \le 2$ nm^2 and $5 \le L \le 30$ nm, respectively.

It is known that vibration frequency of structure is a function of amplitude of vibration in a nonlinear problem (Mousavi and Fariborz 2012). Therefore, Eqs. (21) and (30) can be written as

$$\lambda_{nl} = \lambda + \beta A^2 \tag{33}$$

According to Eq. (33) the non-linear frequencies have a parabolic relation with a maximum amplitude of vibration. In this relation β can be defined as the nonlinear correction coefficient.

The nonlinear correction coefficient for the C-C and F-F boundary conditions can be obtained from Eq. (21)

$$\begin{aligned} &\beta_1 = \frac{9\pi^4}{32\alpha}, \beta_2 = \frac{144\pi^4}{32\alpha}, \quad \beta_3 = \frac{972\pi^4}{32\alpha}, \\ &\alpha = \left[1 + \mu (n\pi)^2\right] \qquad\qquad n = 1,2,\ldots \end{aligned} \tag{34}$$

And for the C-F boundary conditions from Eq. (30)

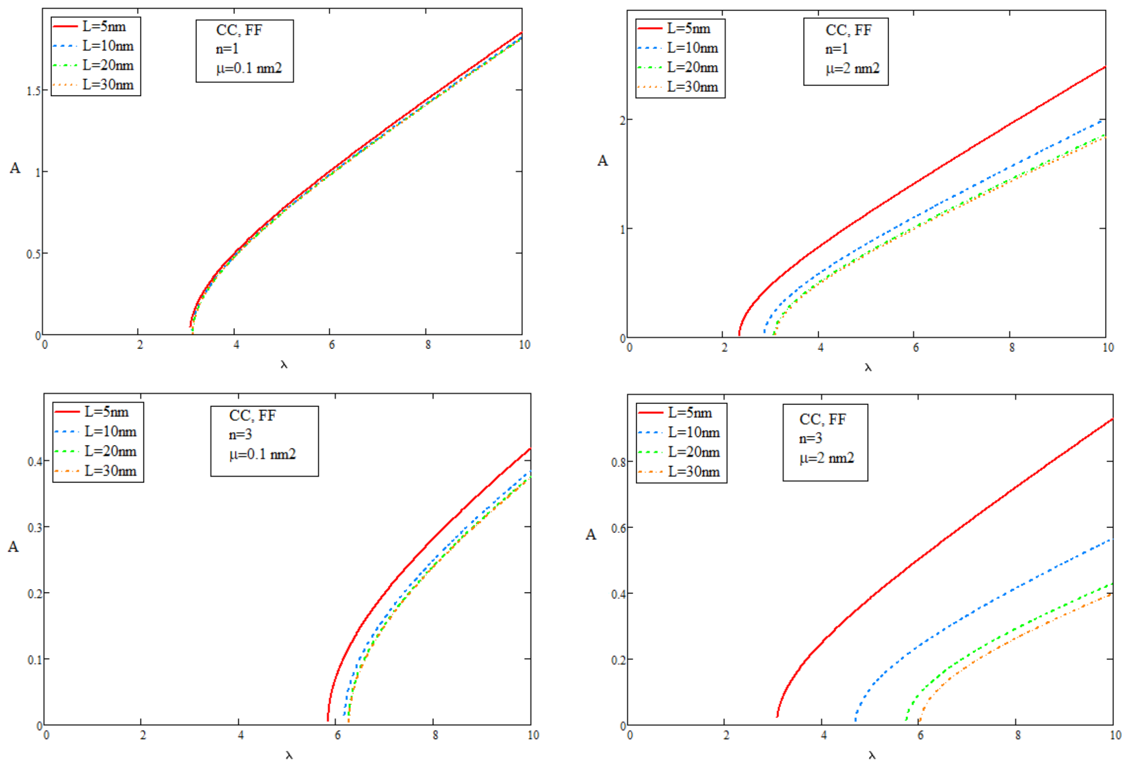

Fig. 1 Backbone curves of nanotubes for CC and FF boundary conditions for the fundamental and the third frequency (L = 5, 10, 20, 30 nm)

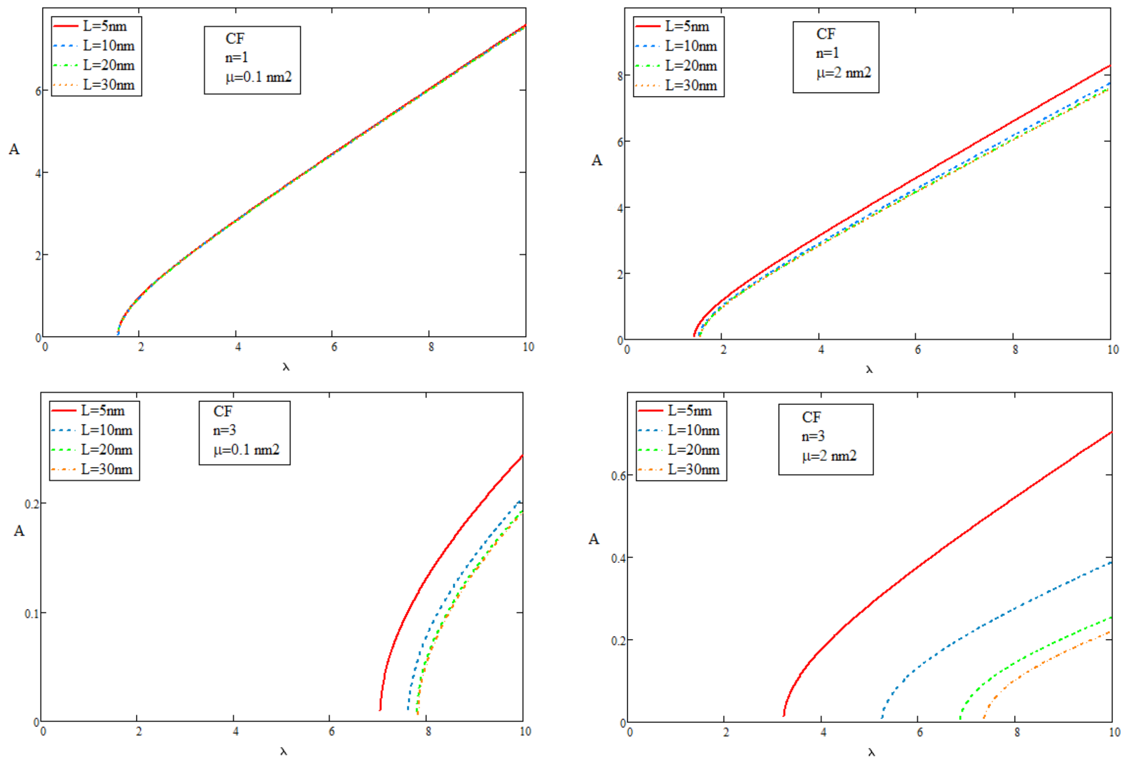

Fig. 2 Backbone curves of nanotubes for CF boundary conditions for the fundamental and the third mode frequency (L=5, 10, 20, 30 nm)

$$\beta_1 = \frac{9\pi^4}{512\alpha}, \beta_2 = \frac{729\pi^4}{512\alpha}, \beta_3 = \frac{5625\pi^4}{512\alpha},$$
$$\alpha = \left[1 + \mu\left(\frac{(2n-1)\pi}{2}\right)^2\right] \quad n = 1,2,... \tag{35}$$

It is seen that correction factor depends on the length L of the nanorod and the nonlocal parameter μ. The backbone curve (the amplitude-frequency relationship) for the above cases are given in Fig. 1-2 for various values of μ and L. It is known that the rod can be considered as the strongly nonlinear system when it vibrates with non-dimensional amplitudes of the order 4. These curves show that nonlinear terms in equation of motion have the hardening effect. When the deflection is zero, linear frequencies are obtained. Points where the amplitudes are zero correspond to linear frequency values. It is seen that bifurcation points are getting farther for lower L and higher μ values due to small scale effects. The amplitudes are greater for nonzero μ and lower L values. Long range interactions make the nanotube softer. The amplitudes of the CF rod are greater than that of CC and FF rod for the same frequency value especially for lower modes.

The nonlocal effects are important, especially for the low L values. And they are lost after a certain L value. This critical L value is smaller for lower modes and getting higher for the higher modes. These values are L = 5, 20 and 30 nm for the first, second and third modes, respectively, for the present problem since the backbone curves for the different μ values are approximately coincident at these L values.

Figs. 3-5 show graphical representation of frequency of internal resonances between the vibration modes. It is seen from the figures that internal resonance exists between first and third modes for CC and FF rod for both local and nonlocal elasticity. There is no internal resonance

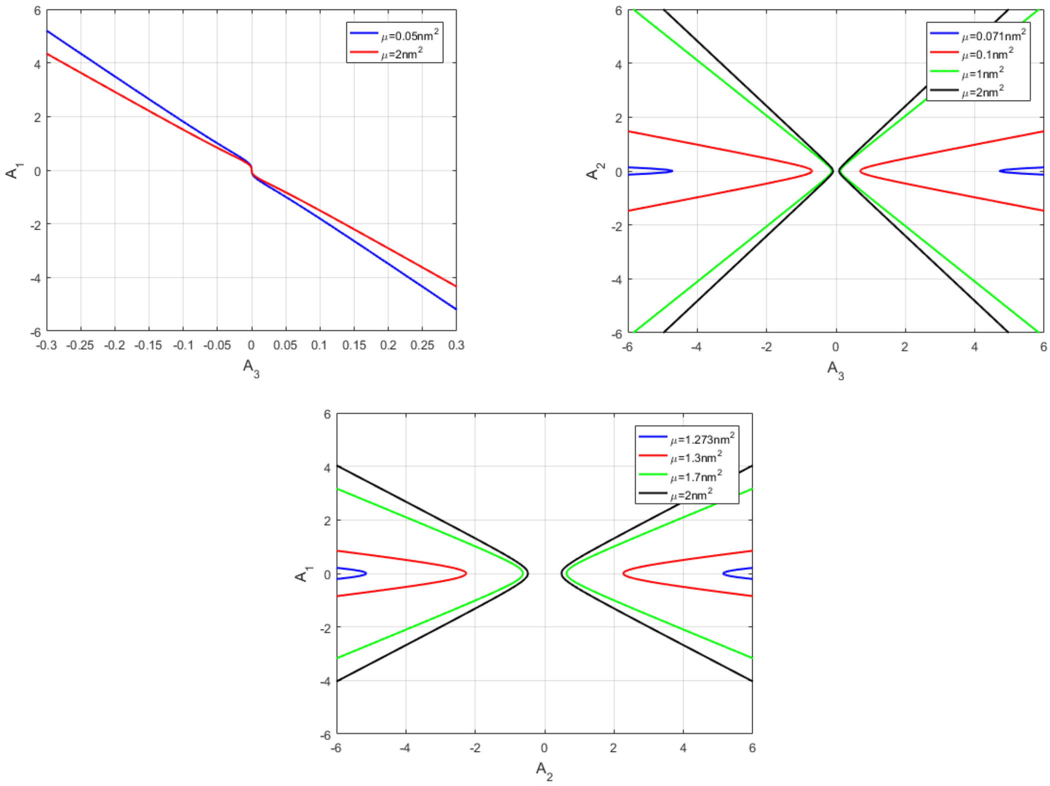

Fig. 3 Internal resonance curves for the first, second and third modes of CC boundary condition (L = 5 nm)

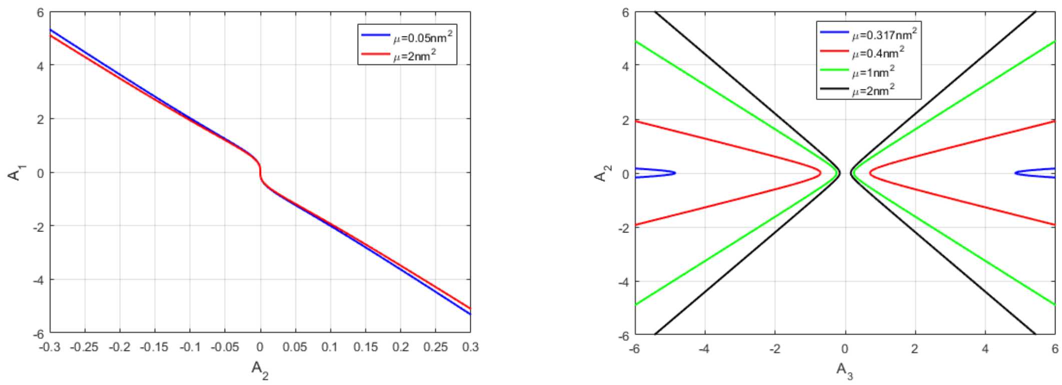

Fig. 4 Internal resonance curves for the first, second and third modes of CF boundary condition (L = 5 nm)

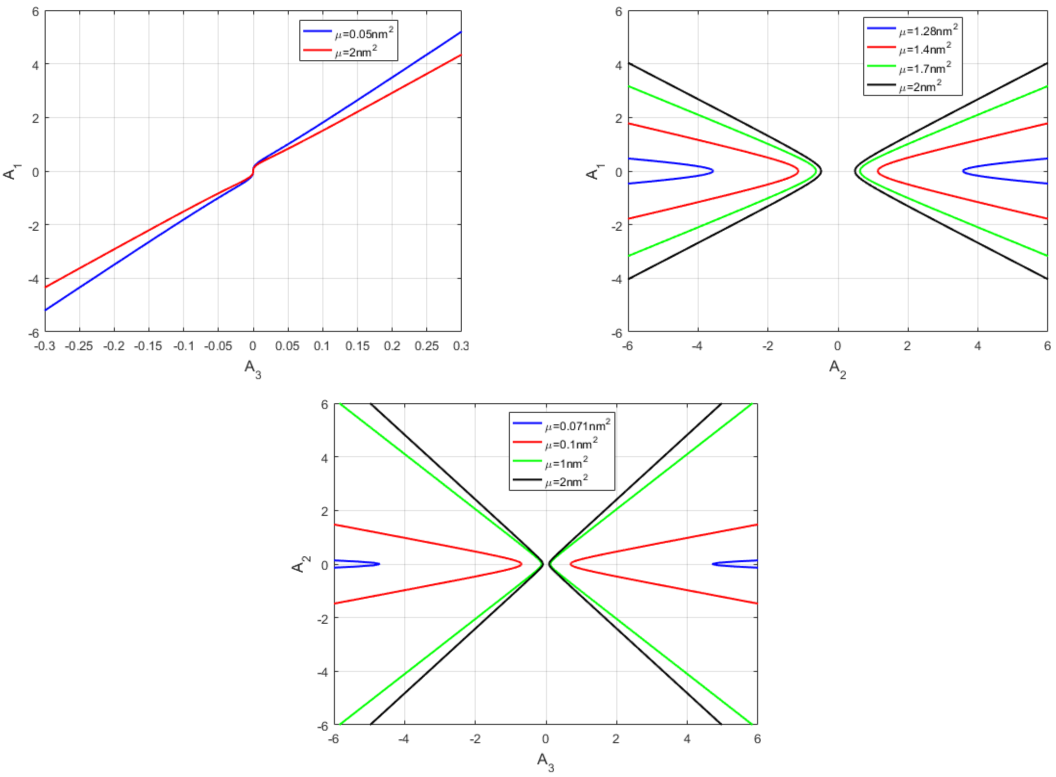

Fig. 5 Internal resonance curves for the first, second and third modes of FF boundary condition (L = 5 nm)

between first and second modes and second and third modes for these boundary conditions for the local elasticity. First and third mode interaction curves pass from the origin with a saddle point. Remaining interactions are more sensitive for nonlocal elasticity. Interaction curves move outward with decreasing nonlocal parameter. For CF boundary condition, there is an internal resonance between first and second modes and second and third modes. Nonlocal effects lead to additional internal modal interaction for nanorod vibration.

4. Conclusions

An elastic finite strain rod model has been developed and applied to investigate the small-scale effect on the nonlinear axial vibrations of nanorods. Nonlocal elasticity theory is used in the formulation and Galerkin method is utilized in the solution. Clamped-clamped, free-free and clamped-free boundary conditions have been considered. Backbone curves are obtained for considering boundary conditions.

The following observations were drawn:

(a) Some additional internal resonance interactions we are observed due to nonlocal effects. There is an internal resonance between first and third modes for CC and FF boundary conditions in the local elasticity solution. However, there are internal resonances between all modes (first-second, first-third and second-third modes) for CC and FF boundary conditions in the nonlocal elasticity formulation. For CF boundary conditions there is an internal resonance between first and second modes for both theories. Moreover, there is an internal resonance between second and third modes due to nonlocal effects for CF boundary conditions.

(b) Higher maximum amplitudes are obtained for lower L values for the same frequency for the nonlocal case.

(c) The nonlocal effects are lost after length L=30nm for the present nanorod problems.

(d) Nonlocal effects lead to additional internal modal interaction for nanorod vibration.

References

Adhikari, S., Murmu, T. and McCarthy, M.A. (2014), “Frequency domain analysis of nonlocal rods embedded in an elastic medium”, *Physica E*: *Low-dimens*. *Syst*. *Nanostruct*., **59**, 33-40.

Ansari, R. and Hemmatnezhad, M. (2011), “Nonlinear vibrations of embedded multi-walled carbon nanotubes using a variational approach”, *Math*. *Comput*. *Model*., **53**(5-6), 927-938.

Aydogdu, M. (2009), “Axial vibration of the nanorods with the nonlocal continuum rod model”, *Physica E*: *Low-dimens*. *Syst*. *Nanostruct*., **41**(5), 861-864.

Aydogdu, M. (2014), “Longitudinal wave propagation in multiwalled carbon nanotubes”, *Compos*. *Struct*., **107**, 578-584.

Aydogdu, M. and Arda, M. (2016), “Forced vibration of nanorods using nonlocal elasticity”, *Adv*. *Nano Res*., *Int*. *J*., **4**(4), 265-279.

Aydogdu, M. and Elishakoff, I. (2014), “On the vibration of nanorods restrained by a linear spring in-span”, *Mech*. *Res*. *Commun*., **57**, 90-96.

Bagdatli, S.M. (2015), “Non-linear transverse vibrations of tensioned nanobeams using nonlocal beam theory”, *Struct*. *Eng*. *Mech*., *Int*. *J*., **55**(2), 281-298.

Besseghier, A., Heireche, H., Bousahla, A.A., Tounsi, A. and Benzair, A. (2015), “Nonlinear vibration properties of a zigzag single-walled carbon nanotube embedded in a polymer matrix”, *Adv*. *Nano Res*., *Int*. *J*., **3**(1), 29-37.

Chaht, F.L., Kaci, A., Houari, M.S.A., Tounsi, A., Bég A.O. and Mahmoud, S.R. (2015), “Bending and buckling analyses of functionally graded material (FGM) size-dependent nanoscale beams including the thickness stretching effect”, *Steel Compos*. *Struct*., *Int*. *J*., **18**(2), 425-442.

Cigeroglu, E. and Samandari, H. (2012), “Nonlinear free vibration of double walled carbon nanotubes by using describing function method with multiple trial functions”, *Physica E*: *Low-dimens*. *Syst*. *Nanostruct*., **46**, 160-173.

Cigeroglu, E. and Samandari, H. (2014), “Nonlinear free vibrations of curved double walled carbon nanotubes using differential quadrature method”, *Physica E*: *Low-dimens*. *Syst*. *Nano Struct*., **64**, 95-105.

Danesh, M., Farajpour, A. and Mohammadi, M. (2012), “Axial vibration analysis of a tapered nanorod based on nonlocal elasticity theory and differential quadrature method”, *Mech*. *Res*. *Commun*., **39**(1), 23-27.

Ebrahimi, F. and Mahmoodi, F. (2018), “Vibration analysis of carbon nanotubes with multiple cracks in thermal environment”, *Adv*. *Nano Res*., *Int*. *J*., **6**(1), 57-80.

Eringen, A.C. (1976), *Nonlocal polar field models*, Academic Press, New York, NY, USA.

Eringen, A.C. (1983), “On differential equations of nonlocal elasticity and solutions of screw dislocation and surface-waves”, *J*. *Appl*. *Phys*., **54**, 4703-4710.

Farokhi, H. and Ghayesh, M.H. (2015), “Thermo-mechanical dynamics of perfect and imperfect Timoshenko microbeams”, *Int*. *J*. *Eng*. *Sci*., **91**, 12-33.

Farokhi, H., Ghayesh, M.H. and Amabili, M. (2013), “Nonlinear dynamics of a geometrically imperfect microbeam based on the modified couple stress theory”, *Int*. *J*. *Eng*. *Sci*., **68**, 11-23.

Fernandes, R., El-Borgi., S., Mousavi, S.M., Reddy, J.N. and Mechmoum, A. (2017), "Nonlinear size-dependent longitudinal vibration of carbon nanotubes embedded in an elastic medium", *Phys. E: Low dimens. Syst. Nanostruct.*, **88**(5), 18-25.

Fu, Y.M., Hong, J.W. and Wang, X.Q. (2006), "Analysis of nonlinear vibration for embedded carbon nanotubes", *J. Sound Vib.*, **296**(4-5), 746-756.

Ghayesh, M.H. (2014), "Nonlinear size dependent behavior of single-walled carbon nanotubes", *Appl. Phys. A*, **117**(3), 1393-1399.

Ghayesh, M.H. and Farokhi, H. (2015a), "Chaotic motion of a parametrically excited microbeam", *Int. J. Eng. Sci.*, **96**, 34-45.

Ghayesh, M.H. andd Farokhi, H. (2015b), "Nonlinear dynamics of microplates", *Int. J. Eng. Sci.*, **86**, 60-73.

Ghayesh, M.H., Amabili, M. and Farokhi, H. (2013a), "Three-dimensional nonlinear size-dependent behaviour of Timoshenko microbeams", *Int. J. Eng. Sci.*, **71**, 1-14.

Ghayesh, M.H., Farokhi, H. and Amabili, M. (2013b), "Nonlinear dynamics of a microscale beam based on the modified couple stress theory", *Compos. Part B*, **50**, 318-324.

Ghayesh, M.H., Farokhi, H. and Amabili, M. (2013c), "Nonlinear behaviour of electrically actuated MEMS resonators", *Int. J. Eng. Sci.*, **71**, 137-155.

Ghayesh, M.H., Amabili, M. and Farokhi, H. (2013d), "Nonlinear forced vibrations of a microbeam based on the strain gradient elasticity theory", *Int. J. Eng. Sci.*, **63**, 52-60.

Ghayesh, M.H., Farokhi, H. and Amabili, M. (2014), "In-plane and out-of-plane motion characteristics of microbeams with modal interactions", *Compos. Part B*, **60**, 423-439.

Ghayesh, M.H., Farokhi, H. and Alici, G. (2016), "Size-dependent performance of microgyroscopes", *Int. J. Eng. Sci.*, **100**, 99-111.

Ghayesh, M.H. (2018a), "Functionally graded microbeams: Simultaneous presence of imperfection and viscoelasticity", *Int. J. Mech. Sci.*, **140**, 339-350.

Ghayesh, M.H. (2018b), "Nonlinear vibration analysis of axially functionally graded shear-deformable tapered beams", *Appl. Math. Model.*, **59**, 583-596.

Ghayesh, M.H. (2018c), "Dynamics of functionally graded viscoelastic microbeams", *Int. J. Eng. Sci.*, **124**,115-131.

Gholipour, A., Farokhi, H. and Ghayesh, M.H. (2015), "In-plane and out-of-plane nonlinear size-dependent dynamics of microplates", *Nonlin. Dyn.*, **79**, 1771-1785.

Hajnayeb, A. and Khadem, S.E. (2015), "An analytical study on the nonlinear vibration of a double-walled carbon nanotube", *Struct. Eng. Mech., Int. J.*, **54**(5), 987-998.

Iijima, S. (1991), "Helical microtubules of graphitic carbon", *Nature*, **354**, 56-58.

Ke, L.L., Xiang, Y., Yang, J. and Kitipornchai, S. (2009), "Nonlinear free vibration of embedded double-walled carbon nanotubes based on nonlocal Timoshenko beam theory", *Computat. Mater. Sci.*, **47**(2), 409-417.

Lai, W.M., Rubin, D. and Krempl, E. (2010), *Introduction to Continuum Mechanics*, Butterworth Heinemann.

Mousavi, S.M. and Fariborz, S.J. (2012), "Free vibration of a rod undergoing finite strain", *J. Phys.: Conference Series*, **382**(1), 012011.

Murmu, T., Adhikari, S. and McCarthy, M.A. (2014), "Axial vibration of embedded nanorods under transverse magnetic field effects via nonlocal elastic continuum theory", *J. Comput. Theor. Nanosci.*, **11**, 1-7.

Peddieson, J., Buchanan, G.R. and McNitt, R.P. (2003), "Application of nonlocal continuum models to Nanotechnology", *Int. J. Eng. Sci.*, **41**(3-5), 305-312.

Pour, H.R., Vossough, H., Beygipoor, M.M.H.G. and Azimzadeh, A. (2015), "Nonlinear vibration analysis of a nonlocal sinusoidal shear deformation carbon nanotube using differential quadrature method", *Struct. Eng. Mech., Int. J.*, **54**(6), 1061-1073.

Rahmani, O., Refaeinejad, V. and Hosseini, S.A.H. (2017), "Assessment of various nonlocal higher order theories for the bending and buckling behavior of functionally graded nanobeams", *Steel Compos. Struct., Int. J.*, **23**(3), 339-350.

Sudak, L.J. (2003), "Column buckling of multiwalled carbon nanotubes using nonlocal continuum Mechanics", *J. Appl. Phys.*, **94**, 7281-7287.

Wang, L. and Hu, H. (2005), "Flexural wave propagation in single-walled carbon nanotubes", *Phys. Rev. B*, **71**, 195412.

Wang, Q. and Wang, C.M. (2007), "The constitutive relation and small scale parameter of nonlocal continuum mechanics for modeling carbon nanotubes", *Nanotechnology*, **18**(7), 075702.

Yang, J., Ke, L.L. and Kitipornchai, S. (2010), "Nonlinear free vibration of single-walled carbon nanotubes using nonlocal Timoshenko beam theory", *Phys. E: Low-dimens. Syst. Nanostruct.*, **42**(5), 1727-1735.

JL