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A third-order parabolic shear deformation beam theory for nonlocal vibration analysis of magneto-electro-elastic nanobeams embedded in two-parameter elastic foundation

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Abstract. This article investigates vibration behavior of magneto-electro-elastic functionally graded (MEE-FG) nanobeams embedded in two-parameter elastic foundation using a third-order parabolic shear deformation beam theory. Material properties of MEE-FG nanobeam are supposed to be variable throughout the thickness based on power-law model. Based on Eringen's nonlocal elasticity theory which captures the small size effects and using the Hamilton's principle, the nonlocal governing equations of motions are derived and then solved analytically. Then the influences of elastic foundation, magnetic potential, external electric voltage, nonlocal parameter, power-law index and slenderness ratio on the frequencies of the embedded MEE-FG nanobeams are studied.

Keywords: magneto-electro-elastic FG nanobeam; free vibration; nonlocal elasticity theory; higher order beam theory

1. Introduction

In the last decade, smart materials have gained notable attention amongst researchers due to their enormous application potentials. In response to external stimuli such as electric and magnetic fields as well as mechanical forces, these materials have characteristics and properties which alter in a controlled fashion. Magneto-electro-elastic materials (MEE) have the ability to transform magnetic, electric and mechanical energies from one form to the others and this makes them beneficial for application in sensing and actuating devices, control of structural vibrations and smart structure technology (Milazzo *et al*. 2009). Free vibration of multiphase and layered MEE beam for $BaTiO_3$–$CoFe_2O_4$ composite is carried out by Annigeri *et al*. (2007). Kumaravel *et al*. (2007) researched linear buckling and free vibration behavior of layered and multiphase MEE beam under thermal environment. Transient dynamic response of multiphase MEE cantilever beam is presented by Daga *et al*. (2009) using finite element method. Razavi and Shooshtari (2015) studied nonlinear free vibration of symmetric MEE laminated rectangular plates with simply

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supported boundary condition. They used the first order shear deformation theory considering the von Karman's nonlinear strains to obtain the equations of motion, whereas Maxwell equations for electrostatics and magneto-statics are used to model the electric and magnetic behavior. Shooshtari and Razavi (2015) investigated nonlinear free vibration of symmetrically laminated MEE rectangular plate resting on an elastic foundation.

When the material properties are assumed to be variable in spatial directions, the structure material should be referred to as functionally graded materials (FGMs). The positive points of the FGMs are interesting enough for the authors to employ them in researches dealing with the mechanical behavior of structures (Ebrahimi *et al*. 2009, 2016b, Ebrahimi and Rastgoo 2009, 2011, Ebrahimi and Barati 2016d, j, m, n, o, 2017a, Ebrahimi and Dabbagh 2016, Ebrahimi and Hosseini 2016a, Ebrahimi 2013). Also functionally graded magneto–electro-elastic materials (MEE-FG) provide a novel combination of both MEEMs and FGMs. Hence, this subject gained considerable attentions in the last decades. Pan and Han (2005) proposed an exact solution for the multilayered rectangular plate made of functionally graded, anisotropic, and linear MEE materials. In this work, they supposed that the edges of the plate are under simply supported conditions, general mechanical, electric and magnetic boundary conditions can be applied on both the top and bottom surfaces of the plate. Huang *et al*. (2007) researched the plane stress problem of generally anisotropic MEE beams with the coefficients of elastic compliance, piezoelectricity, dielectric impermeability, piezo-magnetism, magnetoelectricity, and magnetic permeability being arbitrary functions of the thickness coordinate. In another study, three-dimensional (3D) static behavior of doubly curved MEE-FG shells under the mechanical load, electric displacement and magnetic flux using an asymptotic approach is investigated by Wu and Tsai (2007). Also Ebrahimi and Barati (Ebrahimi and Barati 2016b, g, h, i, k, 2017b, c, Ebrahimi and Salari 2015a, 2016, Ebrahimi *et al*. 2015, Ebrahimi and Mokhtari 2014) investigated mechanical behavior of piezoelectric functionally graded beams and plates via nonlocal elasticity theory. Li *et al*. (2008) investigated the problem of a functionally graded, transversely isotropic, MEE circular plate acted on by a uniform load. Recently, Wang *et al*. (2015) investigated the static behavior of a functionally graded circular plate made of MEE materials under tension and bending. Kattimani and Ray (2015) analyzed active control of geometrically nonlinear vibrations of MEE-FG plates.

It is noticed from the experimental data that small-size structures such as MEE micro/nano structures are different from typical ones due to their size influences. So, classical continuum theories are unable to describe the size-dependent behavior of micro/nano structures. To remedy the drawbacks of the classical continuum theories for the size-dependency of these structures, several higher order continuum theories with new material parameters are suggested. Eringen's nonlocal elasticity theory is one of the nonlocal continuum theories to analysis of nanostructures. To this purpose, Ke and Wang (2014) investigated the free vibration of MEE nanobeams based on the nonlocal theory and Timoshenko beam theory. They supposed that the MEE nanobeam is subjected to the external electric potential, magnetic potential and uniform temperature rise. In another study, Ke *et al*. (2014) investigated the free vibration behavior of MEE nanoplates based on the nonlocal theory and Kirchhoff plate theory. Li *et al*. (2014) analyzed buckling and free vibration of MEE nanoplate resting on Pasternak foundation based on nonlocal Mindlin theory. Ansari *et al*. (2015) studied forced vibration behavior of higher order shear deformable magneto–electro–thermo elastic (METE) nanobeams based on the nonlocal elasticity theory in conjunction with the von Kármán geometric nonlinearity. Wu *et al*. (2015) studied surface effects on anti-plane shear waves propagating in MEE nanoplates. As seen, there is no study investigating the small scale influence on free vibration analyses of FG-MEE nanobeams, while it is necessary to be familiar with the mechanical behavior of MEE-FG nanoscale structures for nano/micro-electromechnaical systems (NEMS/MEMS) fabrication. Without the influences of magnetic and electric fields only a few works are published. Among them, Şimşek and Yurtcu (2013) presented analytical solutions for bending and buckling of functionally graded nanobeams based on the nonlocal Timoshenko beam ttheory. Rahmani and Pedram (2014) analyzed the size effect on vibration of functionally graded nanobeams based on nonlocal Timoshenko beam theory. Also, Ebrahimi and Salari (2015b, c) studied thermo-mechanical behavior of nonlocal temperature-dependent FG nanobeams. Ebrahimi and Barati (2015) presented a nonlocal higher-order shear deformation beam theory for vibration analysis of size-dependent FG nanobeams. Ebrahimi and Barati (2016e) carried out dynamic modeling of a thermo–piezo-electrically actuated nanosize beam subjected to a magnetic field. Ebrahimi and Barati (2016c) presented buckling analysis of nonlocal third-order shear deformable functionally graded piezoelectric nanobeams embedded in elastic medium. Ebrahimi and Barati (2016f) examined electromechanical buckling behavior of smart piezoelectrically actuated higher-order size-dependent graded nanoscale beams in thermal environment. Ebrahimi *et al*. (Ebrahimi *et al*. 2016k, Ebrahimi and Zia 2015, Ebrahimi *et al*. 2016a, Ebrahimi and Barati 2016a, d, l, Ebrahimi and Hosseini 2016a, b, c, Ebrahimi and Habibi 2016) investigated various effects on hygro-thermo-mechanical vibration and buckling behavior of nanostructures.

The main purpose of this article is to investigate the free vibration behavior of MEE-FG nanobeams embedded in elastic foundation using a higher order shear deformation beam theory in which shear deformation effect is involved without the need for shear correction factors. To include the surrounding elastic medium, the Winkler and Pasternak elastic foundation models are utilized, including linear and shear deformation of the elastic medium, respectively. The magneto-electro-mechanical material properties of the beam are supposed to be graded in the thickness direction according to the power law distribution. Based on Eringen's nonlocal elasticity theory, the small size influence is considered. Governing equations for the free vibration responses of a MEE-FG nanobeam on elastic foundation are derived via Hamilton's principle and solved using analytical method. Some numerical examples indicate the influences of elastic foundation, magnetic potential, external electric voltage, power-law index and nonlocal parameter on vibration responses of embedded MEE-FG nanobeams.

2. Theoretical formulations

2.1 The material properties of MEE-FG nanobeams

Assume a MEE-FG nanobeam composed of $BaTiO_3$ and $CoFe_2O_4$ materials exposed to a magnetic potential Υ (x, z, t) and electric potential Φ (x, z, t), with length L and uniform thickness h, as shown in Fig. 1. The effective material properties of the MEE-FG nanobeam are supposed to change continuously in the z-axis direction (thickness direction) based on the power-law model. So, the effective material properties, P, can be stated in the following form

$$P = P_2 V_2 + P_1 V_1 \tag{1}$$

In which P_1 and P_2 denote the material properties of the bottom and higher surfaces, respectively. Also V_1 and V_2 are the corresponding volume fractions related by

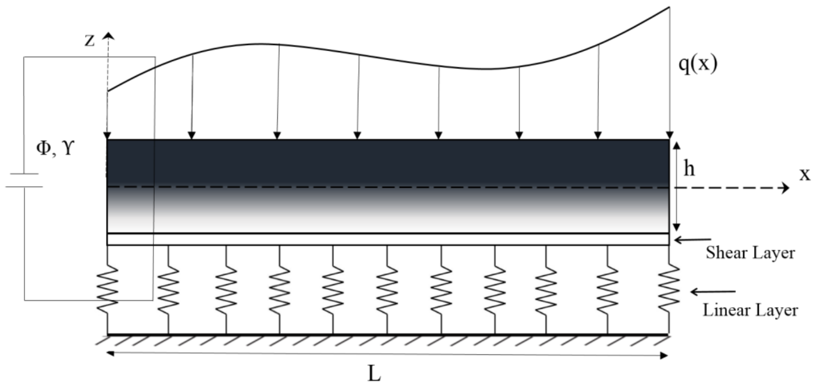


Fig. 1 Configuration of a MEE-FG nanobeam

$$V_2 = (\frac{z}{h} + \frac{1}{2})^p, \quad V_1 = 1 - V_2 \tag{2}$$

Therefore according to Eqs. (1) and (2), the effective magneto-electro-elastic material properties of the FG beam is defined as

$$P(z) = \left(P_2 - P_1\right)\left(\frac{z}{h} + \frac{1}{2}\right)^p + P_1 \tag{3}$$

where p is power-law exponent which is non-negative and estimates the material distribution through the thickness of the nanobeam and z is the distance from the mid-plane of the graded piezoelectric beam. It must be noted that, the top surface at $z = +h/2$ of MEE-FG nanobeam is assumed $CoFe_2O_4$ rich, whereas the bottom surface ($z = -h/2$) is $BaTiO_3$ rich.

2.2 Nonlocal elasticity theory for the MEE-FG *materials*

Contrary to the constitutive equation of classical elasticity theory, Eringen's nonlocal theory (Eringen 1983) notes that the stress state at a point inside a body is regarded to be function of strains of all points in the neighbor regions. For a nonlocal magneto-electro-elastic solid the basic equations with zero body force may be defined as Ke and Wang (2014)

$$\sigma_{ij} = \int_V \alpha\left(\left|x' - x\right|, \tau\right)\left[C_{ijkl}\varepsilon_{kl}(x') - e_{mij}E_m(x') - q_{nij}H_n(x')\right]dV(x') \tag{4a}$$

$$D_i = \int_V \alpha\left(\left|x' - x\right|, \tau\right)\left[e_{ikl}\varepsilon_{kl}(x') + s_{im}E_m(x') + d_{in}H_n(x')\right]dV(x') \tag{4b}$$

$$B_i = \int_V \alpha\left(\left|x' - x\right|, \tau\right)\left[q_{ikl}\varepsilon_{kl}(x') + d_{im}E_m(x') + \chi_{in}H_n(x')\right]dV(x') \tag{4c}$$

where σ_{ij}, ε_{ij}, D_i, E_i, B_i and H_i denote the stress, strain, electric displacement, electric field components, magnetic induction and magnetic field and displacement components, respectively;

C_{ijkl}, e_{mij}, s_{im}, q_{nij}, d_{ij} and χ_{ij} are the elastic, piezoelectric, dielectric constants, piezomagnetic, magnetoelectric, magnetic constants, respectively; α ($|x' - x|$, τ) is the nonlocal kernel function and $|x' - x|$ is the Euclidean distance. $\tau = e_0 a/l$ is defined as scale coefficient, where e_0 is a material constant which is determined experimentally or approximated by matching the dispersion curves of plane waves with those of atomic lattice dynamics; and a and l are the internal and external characteristic length of the nanostructures, respectively. Finally it is possible to represent the integral constitutive relations given by Eq. (4) in an equivalent differential form as

$$\sigma_{ij} - (e_0\, a)^2 \nabla^2 \sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{mij}E_m - q_{nij}H_n \tag{5a}$$

$$D_i - (e_0\, a)^2 \nabla^2 D_i = e_{ikl}\varepsilon_{kl} + s_{im}E_m + d_{in}H_n \tag{5b}$$

$$B_i - (e_0\, a)^2 \nabla^2 B_i = q_{ikl}\varepsilon_{kl} + d_{im}E_m + \chi_{in}H_n \tag{5c}$$

where ∇^2 is the Laplacian operator and $e_0 a$ is the nonlocal parameter revealing the size influence on the response of nanostructures Ebrahimi and Barati (2015).

2.3 Nonlocal MEE-FG nanobeam model

Based on parabolic third order beam theory, the displacement field at any point of the beam is supposed to be in the form Wu and Tsai (2007)

$$u_x(x,z) = u(x) + z\psi(x) - \alpha z^3 (\psi + \frac{\partial w}{\partial x}) \tag{6a}$$

$$u_z(x,z) = w(x) \tag{6b}$$

in which $\alpha = 4/3h^2$ and u and w are displacement components in the mid-plane along the coordinates x and z, respectively, while ψ denotes the total bending rotation of the cross-section about y axis. To satisfy Maxwell's equation in the quasi-static approximation, the distribution of electric and magnetic potential along the thickness direction is supposed to change as a combination of a cosine and linear variation as follows Ke and Wang (2014)

$$\Phi(x,z,t) = -\cos(\xi z)\phi(x,t) + \frac{2z}{h}V \tag{7a}$$

$$\Upsilon(x,z,t) = -\cos(\xi z)\gamma(x,t) + \frac{2z}{h}\Omega \tag{7b}$$

where $\xi = \pi / h$. Also, V and Ω are the initial external electric voltage and magnetic potential applied to the MEE-FG nanobeam. Considering strain–displacement relationships on the basis of parabolic beam theory, the non-zero strains can be stated as

$$\varepsilon_{xx} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} + z^3\varepsilon_{xx}^{(3)} \tag{8}$$

$$\gamma_{xz} = \gamma_{xz}{}^{(0)} + z^2 \gamma_{xz}{}^{(2)} \tag{9}$$

where

$$\varepsilon_{\mathrm{xx}}{}^{(0)} = \frac{\partial u}{\partial x},\ \varepsilon_{\mathrm{xx}}{}^{(1)} = \frac{\partial \psi}{\partial x},\ \varepsilon_{\mathrm{xx}}{}^{(3)} = -\alpha\left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) \tag{10}$$

$$\gamma_{xz}{}^{(0)} = \frac{\partial w}{\partial x} + \psi,\ \gamma_{xz}{}^{(2)} = -\beta\left(\frac{\partial w}{\partial x} + \psi\right) \tag{11}$$

And $\beta = \dfrac{4}{h^2}$. According to the Eq. (7), the non-zero components of electric and magnetic field (E_x, E_z, H_x, H_z) can be obtained as

$$E_x = -\Phi_{,x} = \cos(\xi z)\frac{\partial \phi}{\partial x}, \quad E_z = -\Phi_{,z} = -\xi \sin(\xi z)\phi - \frac{2V}{h} \tag{12a}$$

$$H_x = -\Upsilon_{,x} = \cos(\xi z)\frac{\partial \gamma}{\partial x}, \quad H_z = -\Upsilon_{,z} = -\xi \sin(\xi z)\gamma - \frac{2\Omega}{h} \tag{12b}$$

The Hamilton's principle can be stated in the following form to obtain the governing equations of motion

$$\int_0^t \delta(\Pi_S - \Pi_K + \Pi_W)\,dt = 0 \tag{13}$$

where $\prod_S$ is strain energy, $\prod_K$ is kinetic energy and $\prod_W$ is work done by external applied forces. The first variation of strain energy $\prod_S$ can be calculated as

$$\delta \Pi_S = \int_0^L \int_A \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{xz}\delta\gamma_{xz} - D_x\delta E_x - D_z\delta E_z - B_x\delta H_x - B_z\delta H_z\right) dA dx \tag{14}$$

Substituting Eqs. (8) and (9) into Eq. (14) yields

$$\begin{aligned}\delta \Pi_S &== \int_0^L (N\delta\varepsilon_{\mathrm{xx}}{}^{(0)} + M\delta\varepsilon_{\mathrm{xx}}{}^{(1)} + P\delta\varepsilon_{\mathrm{xx}}{}^{(3)} + Q\delta\gamma_{xz}{}^{(0)} + R\delta\gamma_{xz}{}^{(2)})\,dx \\ &+ \int_0^L \int_A \left(-D_x \cos(\xi z)\delta\left(\frac{\partial \phi}{\partial x}\right) + D_z \xi \sin(\xi z)\delta\phi - B_x \cos(\xi z)\delta\left(\frac{\partial \gamma}{\partial x}\right) + B_z \xi \sin(\xi z)\delta\gamma\right) dA dx\end{aligned} \tag{15}$$

in which N, M and Q are the axial force, bending moment and shear force resultants, respectively. Relations between the stress resultants and stress component used in Eq. (15) are defined as

$$\begin{aligned}&N = \int_{-h/2}^{+h/2} \sigma_{xx} dz,\ M = \int_{-h/2}^{+h/2} \sigma_{xx} z dz,\ P = \int_{-h/2}^{+h/2} \sigma_{xx} z^3 dz \\ &Q = \int_{-h/2}^{+h/2} \sigma_{xz} dz,\ R = \int_{-h/2}^{+h/2} \sigma_{xz} z^2 dz\end{aligned} \tag{16}$$

The kinetic energy $\prod_K$ for graded piezoelectric nanobeam is formulated as

$$\Pi_K = \frac{1}{2}\int_0^L \int_A \rho \left((\frac{\partial u_x}{\partial t})^2 + (\frac{\partial u_z}{\partial t})^2 \right) dA dx \tag{17}$$

where ρ is the mass density. The first variation of the kinetic energy is presented as

$$\begin{aligned}\Pi_K = \int_0^L & I_0 (\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t}) + I_1 (\frac{\partial u}{\partial t}\frac{\partial \delta \psi}{\partial t} + \frac{\partial \psi}{\partial t}\frac{\partial \delta u}{\partial t}) + I_2 \frac{\partial \psi}{\partial t}\frac{\partial \delta \psi}{\partial t} \\ & + \alpha[-I_3 \frac{\partial u}{\partial t}(\frac{\partial^2 \delta w}{\partial x \partial t} + \frac{\partial \delta \psi}{\partial t}) - I_3 \frac{\partial \delta u}{\partial t}(\frac{\partial^2 w}{\partial x \partial t} + \frac{\partial \psi}{\partial t}) - I_4 \frac{\partial \psi}{\partial t}(\frac{\partial \delta \psi}{\partial t} + \frac{\partial^2 \delta w}{\partial x \partial t}) \\ & - I_4 \frac{\partial \delta \psi}{\partial t}(\frac{\partial \psi}{\partial t} + \frac{\partial^2 w}{\partial x \partial t}) + \alpha I_6 (\frac{\partial \psi}{\partial t} + \frac{\partial^2 w}{\partial x \partial t})(\frac{\partial \delta \psi}{\partial t} + \frac{\partial^2 \delta w}{\partial x \partial t})] dA dx\end{aligned} \tag{18}$$

In which I_0, I_1, I_2, I_3, I_4 and I_6 are mass inertia and defined as

$$(I_0, I_1, I_2, I_3, I_4, I_6) = \int_A (1, z, z^2, z^3, z^4, z^6)\rho dA \tag{19}$$

It is noticed from Eq. (19), for homogeneous nanobeams, $I_1 = I_3 = 0$.
The work done due to external electric voltage, $\prod_W$, can be written in the form:

$$\begin{aligned}\Pi_W = \int_0^L & (N_H + N_E)\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + q\,\delta w + f\,\delta u - N\delta\varepsilon_{xx}{}^{(0)} - \hat{M}\frac{\partial \delta \psi}{\partial x} \\ & + \alpha P \frac{\partial^2 \delta w}{\partial x^2} - \hat{Q}\delta\gamma_{xz}{}^{(0)} - k_w \delta w + k_p \frac{\partial^2 \delta w}{\partial x^2}) dx\end{aligned} \tag{20}$$

where $\hat{M} = M - \alpha P$, $\hat{Q} = Q - \beta R$ and $q(x)$ and $f(x)$ are the transverse and axial distributed loads and k_w and k_p are foundation parameters and also N_H and N_E are the normal forces induced by magnetic potential and external electric voltage, respectively which are defined as

$$N_E = -\int_A e_{31}\frac{2V}{h} dA,\ N_H = -\int_A q_{31}\frac{2\Omega}{h} dA \tag{21}$$

For a MEE-FG nanobeam in the one dimensional case, the nonlocal constitutive relations (5a)-(5c) may be rewritten as

$$\sigma_{xx} - (e_0 a)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = c_{11}\varepsilon_{xx} - e_{31}E_z - q_{31}H_z \tag{22}$$

$$\sigma_{xz} - (e_0 a)^2 \frac{\partial^2 \sigma_{xz}}{\partial x^2} = c_{55}\gamma_{xz} - e_{15}E_x - q_{15}H_x \tag{23}$$

$$D_x - (e_0 a)^2 \frac{\partial^2 D_x}{\partial x^2} = e_{15}\gamma_{xz} + s_{11}E_x + d_{11}H_x \tag{24}$$

$$D_z - (e_0 a)^2 \frac{\partial^2 D_z}{\partial x^2} = e_{31}\varepsilon_{xx} + s_{33}E_z + d_{33}H_z \tag{25}$$

$$B_x - (e_0 a)^2 \frac{\partial^2 B_x}{\partial x^2} = q_{15}\gamma_{xz} + d_{11}E_x + \chi_{11}H_x \tag{26}$$

$$B_z - (e_0 a)^2 \frac{\partial^2 B_z}{\partial x^2} = q_{31}\varepsilon_{xx} + d_{33}E_z + \chi_{33}H_z \tag{27}$$

Inserting Eqs. (15), (18) and (20) in Eq. (13) and integrating by parts, and gathering the coefficients of δu, δw, $\delta\psi$, $\delta\phi$ and $\delta\gamma$ the following governing equations are obtained

$$\frac{\partial N}{\partial x} + f - I_0 \frac{\partial^2 u}{\partial t^2} - \hat{I}_1 \frac{\partial^2 \psi}{\partial t^2} + \alpha I_3 \frac{\partial^3 w}{\partial x \partial t^2} = 0 \tag{28}$$

$$\frac{\partial \hat{M}}{\partial x} - \hat{Q} - \hat{I}_1 \frac{\partial^2 u}{\partial t^2} - \hat{I}_2 \frac{\partial^2 \psi}{\partial t^2} + \alpha \hat{I}_4 \left(\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^3 w}{\partial x \partial t^2}\right) = 0 \tag{29}$$

$$\begin{aligned} &\frac{\partial Q}{\partial x} + q - (N_H + N_E)\frac{\partial^2 w}{\partial x^2} + \alpha \frac{\partial^2 P}{\partial x^2} - I_0 \frac{\partial^2 w}{\partial t^2} - \alpha I_3 \frac{\partial^3 u}{\partial x \partial t^2} - \alpha I_4 \frac{\partial^3 \psi}{\partial x \partial t^2} \\ &+\alpha^2 I_6 \left(\frac{\partial^3 \psi}{\partial x \partial t^2} + \frac{\partial^4 w}{\partial x^2 \partial t^2}\right) - k_w w + k_p \frac{\partial^2 w}{\partial x^2} = 0 \end{aligned} \tag{30}$$

$$\int_A \left(\cos(\xi z)\frac{\partial D_x}{\partial x} + \xi \sin(\xi z) D_z\right) dA = 0 \tag{31}$$

$$\int_A \left(\cos(\xi z)\frac{\partial B_x}{\partial x} + \xi \sin(\xi z) B_z\right) dA = 0 \tag{32}$$

where $\hat{I}_1 = I_1 - c_1 I_3$, $\hat{I}_2 = I_2 - c_1 I_4$, $\hat{I}_4 = I_4 - c_1 I_6$. Furthermore, the corresponding natural and essential boundary conditions are defined at $x = 0$ and $x = L$ as follows

$$N = 0 \quad \text{or} \quad u = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \tag{33a}$$

$$N\frac{\partial w}{\partial x} + c_1 \frac{\partial P}{\partial x} + \hat{Q} = 0 \quad \text{or} \quad w = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \tag{33b}$$

$$\alpha P = 0 \quad \text{or} \quad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \tag{33c}$$

$$\hat{M} = 0 \quad \text{or} \quad \psi = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \tag{33d}$$

$$\int_A D_x \cos(\beta z)\, dA = 0 \quad \text{or} \quad \phi = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \tag{33e}$$

$$\int_A B_x \cos(\beta z)\, dA = 0 \quad \text{or} \quad \gamma = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \tag{33f}$$

By integrating Eqs. (22)-(27), over the beam's cross-section area, the force-strain and the moment-strain of the nonlocal third order Reddy FG beam theory can be obtained as presented in Appendix. Now we use $\hat{M}$ and $\hat{Q}$ from Eqs. (A20) and (A22) and the identity

$$\alpha \frac{\partial^2}{\partial x^2}(P - \mu \frac{\partial^2 P}{\partial x^2}) = \alpha(E_{xx}\frac{\partial^3 u}{\partial x^3} + F_{xx}\frac{\partial^3 \psi}{\partial x^3} - \alpha H_{xx}(\frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^4 w}{\partial x^4}) + F_{31}^e \frac{\partial^2 \phi}{\partial x^2} + F_{31}^m \frac{\partial^2 \gamma}{\partial x^2}) \tag{34}$$

It must be cited that inserting Eqs. (31) and (32) into Eqs. (A6)-(A9), does not provide an explicit expressions for D_x and D_z. To overcome this problem, by using Eqs. (A6)-(A9), Eqs. (31) and (32) can be re-expressed in terms of u, w, ψ and ϕ. Finally, based on third-order beam theory, the nonlocal equations of motion for a FG-MEE nanobeam can be obtained by substituting for N, $\hat{M}$ and $\hat{Q}$ from Eqs. (A19), (A20) and (A22) into Eqs. (A1)-(A5) as follows

$$\begin{aligned} & A_{xx}\frac{\partial^2 u}{\partial x^2} + K_{xx}\frac{\partial^2 \psi}{\partial x^2} - \alpha E_{xx}\frac{\partial^3 w}{\partial x^3} + A_{31}^e \frac{\partial \phi}{\partial x} + A_{31}^m \frac{\partial \Upsilon}{\partial x} + \mu(-\frac{\partial^2 f}{\partial x^2} + I_0 \frac{\partial^4 u}{\partial x^2 \partial t^2} + I_1 \frac{\partial^4 \psi}{\partial x^2 \partial t^2} \\ & -\alpha I_3 \frac{\partial^4 \psi}{\partial x^2 \partial t^2} - \alpha I_3 \frac{\partial^5 w}{\partial x^3 \partial t^2}) + f - I_0 \frac{\partial^2 u}{\partial t^2} - \hat{I}_1 \frac{\partial^2 \psi}{\partial t^2} + \alpha I_3 \frac{\partial^3 w}{\partial x \partial t^2} = 0 \end{aligned} \tag{35}$$

$$\begin{aligned} & K_{xx}\frac{\partial^2 u}{\partial x^2} + I_{xx}\frac{\partial^2 \psi}{\partial x^2} - \alpha J_{xx}(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^3 w}{\partial x^3}) - \overline{A}_{xz}\left(\psi + \frac{\partial w}{\partial x}\right) + (E_{31}^e - \alpha F_{31}^e)\frac{\partial \phi}{\partial x} + (E_{31}^m - \alpha F_{31}^m)\frac{\partial \gamma}{\partial x} \\ & -\hat{I}_2 \frac{\partial^2 \psi}{\partial t^2} - \hat{I}_1 \frac{\partial^2 u}{\partial t^2} + \alpha \hat{I}_4(\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^3 w}{\partial x \partial t^2}) + (E_{15}^e - \beta F_{15}^e)\frac{\partial \phi}{\partial x} + (E_{15}^m - \beta F_{15}^m)\frac{\partial \gamma}{\partial x} + \mu(\hat{I}_1 \frac{\partial^4 u}{\partial x^2 \partial t^2} \\ & +\hat{I}_2 \frac{\partial^4 \psi}{\partial x^2 \partial t^2} - \alpha \hat{I}_4(\frac{\partial^4 \psi}{\partial x^2 \partial t^2} + \frac{\partial^5 w}{\partial x^3 \partial t^2})) = 0 \end{aligned} \tag{36}$$

$$\begin{aligned} & \overline{A}_{xz}\left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + \mu((N_E + N_M)\frac{\partial^4 w}{\partial x^4} - \frac{\partial^2 q}{\partial x^2} + k_w \frac{\partial^2 w}{\partial x^2} - k_p \frac{\partial^4 w}{\partial x^4}) - (N_E + N_M)\frac{\partial^2 w}{\partial x^2} - (E_{15}^e - \beta F_{15}^e)\frac{\partial^2 \phi}{\partial x^2} \\ & -(E_{15}^m - \beta F_{15}^m)\frac{\partial^2 \gamma}{\partial x^2} - I_0 \frac{\partial^2 w}{\partial t^2} + \alpha(E_{xx}\frac{\partial^3 u}{\partial x^3} + J_{xx}\frac{\partial^3 \psi}{\partial x^3} - \alpha H_{xx}\frac{\partial^4 w}{\partial x^4} + F_{31}^e \frac{\partial^2 \phi}{\partial x^2} + F_{31}^m \frac{\partial^2 \gamma}{\partial x^2}) - k_w w + k_p \frac{\partial^2 w}{\partial x^2} \\ & -\alpha I_3 \frac{\partial^3 u}{\partial x \partial t^2} - \alpha I_4 \frac{\partial^3 \psi}{\partial t^2 \partial x} + \alpha^2 I_6(\frac{\partial^3 \psi}{\partial t^2 \partial x} + \frac{\partial^4 w}{\partial t^2 \partial x^2}) + q + \mu(I_0 \frac{\partial^4 w}{\partial x^2 \partial t^2} + \alpha I_3 \frac{\partial^5 u}{\partial x^3 \partial t^2} + \alpha I_4 \frac{\partial^5 \psi}{\partial t^2 \partial x^3} \\ & -\alpha^2 I_6(\frac{\partial^5 \psi}{\partial t^2 \partial x^3} + \frac{\partial^6 w}{\partial t^2 \partial x^4})) = 0 \end{aligned} \tag{37}$$

$$\begin{aligned} & (E_{15}^e - \beta F_{15}^e)(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \psi}{\partial x}) + F_{11}^e \frac{\partial^2 \phi}{\partial x^2} + F_{11}^m \frac{\partial^2 \gamma}{\partial x^2} + A_{31}^e \frac{\partial u}{\partial x} + (E_{31}^e - \alpha F_{31}^e)\frac{\partial \psi}{\partial x} - \alpha F_{31}^e \frac{\partial^2 w}{\partial x^2} \\ & -F_{33}^e \phi - F_{33}^m \gamma = 0 \end{aligned} \tag{38}$$

$$
\begin{aligned}
&(E_{15}^{m}-\beta F_{15}^{m})(\frac{\partial^2 w}{\partial x^2}+\frac{\partial \psi}{\partial x})+F_{11}^{m}\frac{\partial^2 \phi}{\partial x^2}+X_{11}^{m}\frac{\partial^2 \gamma}{\partial x^2}+A_{31}^{m}\frac{\partial u}{\partial x}+(E_{31}^{m}-\alpha F_{31}^{m})\frac{\partial \psi}{\partial x}-\alpha F_{31}^{m}\frac{\partial^2 w}{\partial x^2}\\
&-F_{33}^{m}\phi-X_{33}^{m}\gamma=0
\end{aligned}
\tag{39}
$$

3. Solution procedure

Here, on the basis the Navier method, an analytical solution of the governing equations for free vibration of a simply supported MEE-FG nanobeam is presented. To satisfy governing equations of motion, the displacement variables are adopted to be of the form

$$u(x,t)=\sum_{n=1}^{\infty}U_n\cos(\frac{n\pi}{L}x)e^{i\omega_n t} \tag{40}$$

$$w(x,t)=\sum_{n=1}^{\infty}W_n\sin(\frac{n\pi}{L}x)e^{i\omega_n t} \tag{41}$$

$$\psi(x,t)=\sum_{n=1}^{\infty}\Psi_n\cos(\frac{n\pi}{L}x)e^{i\omega_n t} \tag{42}$$

$$\phi(x,t)=\sum_{n=1}^{\infty}\Phi_n\sin(\frac{n\pi}{L}x)e^{i\omega_n t} \tag{43}$$

$$\gamma(x,t)=\sum_{n=1}^{\infty}\Upsilon_n\sin(\frac{n\pi}{L}x)e^{i\omega_n t} \tag{44}$$

where U_n, W_n, Ψ_n, Φ_n and Υ_n are the unknown Fourier coefficients to be determined for each n value. Using Eqs. (40)-(44) the analytical solution can be obtained from the following equations

$$
\left\{\begin{pmatrix}
k_{1,1} & k_{1,2} & k_{1,3} & k_{1,4} & k_{1,5}\\
k_{2,1} & k_{2,2} & k_{2,3} & k_{2,4} & k_{2,5}\\
k_{3,1} & k_{3,2} & k_{3,3} & k_{3,4} & k_{3,5}\\
k_{4,1} & k_{4,2} & k_{4,3} & k_{4,4} & k_{4,5}\\
k_{5,1} & k_{5,2} & k_{5,3} & k_{5,4} & k_{5,5}
\end{pmatrix}-\bar{\omega}_n^{\,2}\begin{pmatrix}
m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5}\\
m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & m_{2,5}\\
m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & m_{3,5}\\
m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & m_{4,5}\\
m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & m_{5,5}
\end{pmatrix}\right\}\begin{Bmatrix}U_n\\ \Psi_n\\ W_n\\ \Phi_n\\ \Upsilon_n\end{Bmatrix}=0 \tag{45}
$$

where

$$k_{1,1}=-A_{xx}(\frac{n\pi}{L})^2\ ,\ \ k_{1,2}=-K_{xx}(\frac{n\pi}{L})^2,\ \ k_{1,3}=\alpha E_{xx}(\frac{n\pi}{L})^3,\ \ k_{1,4}=-A_{31}^{e}(\frac{n\pi}{L}),k_{1,5}=-A_{31}^{m}(\frac{n\pi}{L})$$

$$k_{2,2}=-I_{xx}(\frac{n\pi}{L})^2+\alpha J_{xx}(\frac{n\pi}{L})^2-\bar{A}_{xz}\ ,\ \ k_{2,3}=-\bar{A}_{xz}(\frac{n\pi}{L})+J_{xx}(\frac{n\pi}{L})^3,\ \ k_{2,4}=-((E_{15}^{e}-\beta F_{15}^{e})+(E_{31}^{e}-\alpha F_{31}^{e}))(\frac{n\pi}{L}),$$

$$k_{2,5}=-((E_{15}^{m}-\beta F_{15}^{m})+(E_{31}^{m}-\alpha F_{31}^{m}))(\frac{n\pi}{L}),\ k_{3,5}=-((E_{15}^{m}-\beta F_{15}^{m})-\alpha F_{31}^{m})(\frac{n\pi}{L})^2$$

$$k_{3,3}=(N_H+N_E)(\frac{n\pi}{L})^2(1+\mu(\frac{n\pi}{L})^2)-\bar{A}_{xz}(\frac{n\pi}{L})^2-\alpha^2 H_{xx}(\frac{n\pi}{L})^4+K_p(\frac{n\pi}{L})^2(1+\mu(\frac{n\pi}{L})^2)-K_w(1+\mu(\frac{n\pi}{l})^2),$$

$$k_{3,4}=-((E_{15}^e-\beta F_{15}^e)-\alpha F_{31}^e)(\frac{n\pi}{L})^2,\ k_{4,4}=-(F_{11}^e(\frac{n\pi}{L})^2+F_{33}^e), k_{4,4}=-(F_{11}^m(\frac{n\pi}{L})^2+F_{33}^m)$$

$$m_{1,1}=I_0(1+\mu(\frac{n\pi}{L})^2),\quad m_{1,2}=+\hat{I}_1+\mu\hat{I}_1(\frac{n\pi}{L})^2,\quad m_{1,3}=-\alpha I_3(\frac{n\pi}{L})-\mu\alpha I_3(\frac{n\pi}{L})^3,$$

$$m_{2,2}=+\hat{I}_2-\alpha\hat{I}_4+\mu((\hat{I}_2-\alpha\hat{I}_4)(\frac{n\pi}{L})^2,\quad m_{2,3}=-\alpha\hat{I}_4(\frac{n\pi}{L})-\mu\alpha\hat{I}_4(\frac{n\pi}{L})^3,$$

$$m_{3,3}=+I_0+\alpha^2 I_6(\frac{n\pi}{L})^2+\mu(I_0(\frac{n\pi}{L})^2+\alpha^2 I_6(\frac{n\pi}{L})^4),\ m_{1,4}=m_{2,4}=m_{3,4}=m_{4,4}=m_{1,5}=m_{2,5}=m_{3,5}=m_{4,5}=m_{5,5}=0$$

4. Results and discussion

This section provides some numerical examples for the magneto-electro-elastic free vibration characteristics of MEE-FG nanobeams. To achieve this end, the nonlocal FG beam made of $BaTiO_3$ and $CoFe_2O_4$, with magneto-electro-elastic material properties listed in Table 1, is supposed. The beam geometry has the following dimensions: L (length) = 10 nm and h (thickness) = varied. Also, the following relation is described to calculate the non-dimensional natural frequencies

$$\bar{\omega}=\omega L^2\sqrt{(\frac{\rho A}{c_{11}I})_{\mathrm{CoFe_2O_4}}},K_w=k_w\frac{L^4}{(c_{11}I)_{\mathrm{CoFe_2O_4}}},\ K_p=k_p\frac{L^2}{(c_{11}I)_{\mathrm{CoFe_2O_4}}} \tag{46}$$

In which $I = h^3/12$ is the moment of inertia of the cross section of the beam. To verify present model the frequency results are compared with those of nonlocal FGM Timoshenko beams presented by Rahmani and Pedram (2014), due to the fact that any numerical results for the free

Table 1 Magneto-electro-elastic coefficients of material properties (Pan and Han 2005)

Properties	$BaTiO_3$	$CoFe_2O_4$
c_{11} (GPa)	166	286
c_{55}	43	45.3
e_{31} (Cm^{-2})	-4.4	0
e_{15}	11.6	0
q_{31} (N/Am)	0	580.3
q_{15}	0	550
s_{11} ($10^{-9}C^2m^{-2}N^{-1}$)	11.2	0.08
s_{33}	12.6	0.093
χ_{11} ($10^{-6}Ns^2C^{-2}/2$)	5	-590
χ_{33}	10	157
$d_{11} = d_{33}$	0	0
ρ (kgm^{-3})	5800	5300

Table 2 Comparison of the non-dimensional fundamental frequency for a S-S FG nanobeam with various power-law index (L/h = 20)

μ (nm)^2	p = 0		p = 0.5		p = 1		p = 5	
	TBT (Rahmani and Pedram 2014)	Present RBT	TBT (Rahmani and Pedram 2014)	Present RBT	TBT (Rahmani and Pedram 2014)	Present RBT	TBT (Rahmani and Pedram 2014)	Present RBT
0	9.8296	9.829570	7.7149	7.71546	6.9676	6.967613	5.9172	5.916152
1	9.3777	9.377686	7.3602	7.36078	6.6473	6.647300	5.6452	5.644175
2	8.9829	8.982894	7.0504	7.05090	6.3674	6.367454	5.4075	5.406561
3	8.6341	8.634103	6.7766	6.77714	6.1202	6.120217	5.1975	5.196632
4	8.3230	8.323021	6.5325	6.53296	5.8997	5.899708	5.0103	5.009400

Table 3 Comparison of the non-dimensional fundamental frequency for a MEE nanobeam with various nonlocal parameters

	μ = 0	μ = 0.1	μ = 0.2	μ = 0.3	μ = 0.4
Ke and Wang (2014)	3.7388	3.5670	3.1658	2.7209	2.3281
Present	3.6448	3.4773	3.0863	2.6372	2.2451

Table 4 Variation of the first dimensionless frequency of embedded S-S FG nanobeam for various nonlocal parameter, magnetic potentials and electric voltages (L/h = 20)

(K_w, K_p)	μ		Ω = -0.05			Ω = 0			Ω = +0.05		
			p = 0.2	p = 1	p = 5	p = 0.2	p = 1	p = 5	p = 0.2	p = 1	p = 5
(0,0)	0	V = -5	8.55703	8.31260	8.37748	9.43205	8.84645	8.55259	10.2325	9.34987	8.72418
		V = 0	8.41640	7.88376	7.67745	9.30465	8.44476	7.86815	10.1152	8.97075	8.05433
		V = +5	8.27337	7.43021	6.90683	9.17548	8.02299	7.11820	9.99651	8.57489	7.32348
	1	V = -5	8.08984	7.91790	8.03872	9.01035	8.47664	8.22104	9.84516	9.00077	8.39941
		V = 0	7.94094	7.46642	7.30630	8.87690	8.05654	7.50644	9.72317	8.60629	7.70137
		V = +5	7.78919	6.98583	6.49177	8.74141	7.61329	6.71622	9.59964	8.19284	6.93341
	2	V = -5	7.67792	7.57252	7.74446	8.64241	8.15496	7.93354	9.50958	8.69849	8.11823
		V = 0	7.52087	7.09911	6.98123	8.50319	7.71737	7.19042	9.38323	8.28964	7.39369
		V = +5	7.36046	6.59180	6.12362	8.36165	7.25342	6.36107	9.25516	7.85955	6.58997
(25,5)	0	V = -5	12.0943	11.8301	11.7886	12.7285	12.2111	11.9137	13.3325	12.5806	12.0375
		V = 0	11.9952	11.5328	11.3019	12.6343	11.9234	11.4323	13.2426	11.8140	11.5612
		V = +5	11.8953	11.2277	10.7932	12.5395	11.6284	10.9297	13.1522	11.5163	11.0645
	1	V = -5	11.7684	11.5562	11.5503	12.4192	11.9460	11.6780	13.0375	12.3234	11.8042
		V = 0	11.6666	11.2517	11.0531	12.3227	11.6516	11.1864	12.9457	12.0383	11.3181
		V = +5	11.5638	10.9387	10.5324	12.2255	11.3496	10.6722	12.8531	11.7463	10.8102
	2	V = -5	11.4891	11.3223	11.3475	12.1549	11.7199	11.4774	12.786	12.1044	11.6058
		V = 0	11.3848	11.0113	10.8410	12.0563	11.4197	10.9768	12.6923	11.8140	11.1111
		V = +5	11.2795	10.6913	10.3096	11.9569	11.1115	10.4524	12.5979	11.5163	10.5932

Table 5 Variation of the second dimensionless frequency of embedded S-S FG nanobeam for various nonlocal parameter, magnetic potentials and electric voltages (L/h = 20)

(K_w, K_p)	μ		Ω = -0.05			Ω = 0			Ω = +0.05		
			p = 0.2	p = 1	p = 5	p = 0.2	p = 1	p = 5	p = 0.2	p = 1	p = 5
(0,0)	0	V = -5	35.6083	32.9563	31.4249	36.4770	33.5046	31.6119	37.3256	34.0440	31.7979
		V = 0	35.4747	32.5345	30.7053	36.3466	33.0897	30.8967	37.1982	33.6358	31.0870
		V = +5	35.3406	32.1071	29.9685	36.2158	32.6696	30.1646	37.0703	33.2226	30.3594
	1	V = -5	29.9003	27.8605	26.7829	30.9298	28.5069	27.0022	31.9261	29.1390	27.2197
		V = 0	29.7411	27.3602	25.9349	30.7759	28.0182	26.1613	31.7771	28.6610	26.3857
		V = +5	29.5810	26.8506	25.0582	30.6212	27.5207	25.2924	31.6273	28.1749	25.5245
	2	V = -5	26.1741	24.5567	23.7980	27.3442	25.2877	24.0444	28.4663	25.9981	24.2884
		V = 0	25.9920	23.9876	22.8394	27.1700	24.7354	23.0961	28.2990	25.4613	23.3500
		V = +5	25.8087	23.4046	21.8388	26.9947	24.1705	22.1071	28.1307	24.9128	22.3722
(25,5)	0	V = -5	38.5394	36.0121	34.5274	39.3435	36.5145	34.6977	40.1315	37.0100	34.8672
		V = 0	38.4160	35.6264	33.8738	39.2226	36.1342	34.0474	40.0130	36.6349	34.2201
		V = +5	38.2922	35.2366	33.2074	39.1014	35.7498	33.3844	39.8942	36.2559	33.5606
	1	V = -5	33.3371	31.4159	30.3635	34.2635	31.9905	30.5571	35.1655	32.5550	30.7494
		V = 0	33.1944	30.9730	29.6182	34.1247	31.5558	29.8166	35.0302	32.1279	30.0137
		V = +5	33.0510	30.5238	28.8537	33.9852	31.1149	29.0573	34.8944	31.6950	29.2595
	2	V = -5	30.0403	28.5268	27.7662	31.0652	29.1585	27.9777	32.0573	29.7767	28.1877
		V = 0	29.8818	28.0384	26.9491	30.9119	28.6808	27.1670	31.9088	29.3091	27.3832
		V = +5	29.7225	27.5413	26.1065	30.7579	28.1951	26.3314	31.7597	28.8340	26.5544

vibration of magneto-electro-elastic FG nanobeams based on the nonlocal elasticity theory do not exist yet. In this work, the material selection is performed as follows: E_m = 210 GPa, v_m = 0.3, ρ_m = 7800 kgm^{-3} for Steel and E_c = 390 GPa, v_c = 0.24, ρ_c = 3960 kgm^{-3} for Alumina. Therefore, Table 2 presents the fundamental frequency of S-S FG nanobeams in comparison to those of Rahmani and Pedram (2014). Also, the results of homogeneous MEE nanobeam are validated with those of Ke and Wang (2014). They used Euler-Bernoulli beam model their paper and according to Table 3, a good agreement is observed between our model and results of Ke and Wang (2014).

Tables 4-6 present influences of various parameters such as elastic foundation parameters (k_w, k_p), magnetic potential (Ω), external electric voltage (V), power-law index (p) and nonlocal parameter (μ) on the first three non-dimensional frequencies of the simply supported MEE-FG nanobeams at L/h = 20. As a result, when the nonlocal parameter increases the natural frequencies of FG nanobeam reduces due to the reason that existence of nonlocality provides a more flexible beam structure.

Moreover, it is found that the reduction in higher modes due to nonlocality influence is more significant than lower modes. Another important observation is that elastic foundation shows an increasing influence on the stiffness of the beam. So, when the Winkler and Pasternak foundation parameter increases the dimensionless frequencies of the MEE-FG nanobeam rise. Also, it is seen that for all values of Winkler and Pasternak foundation parameters, negative electric voltages

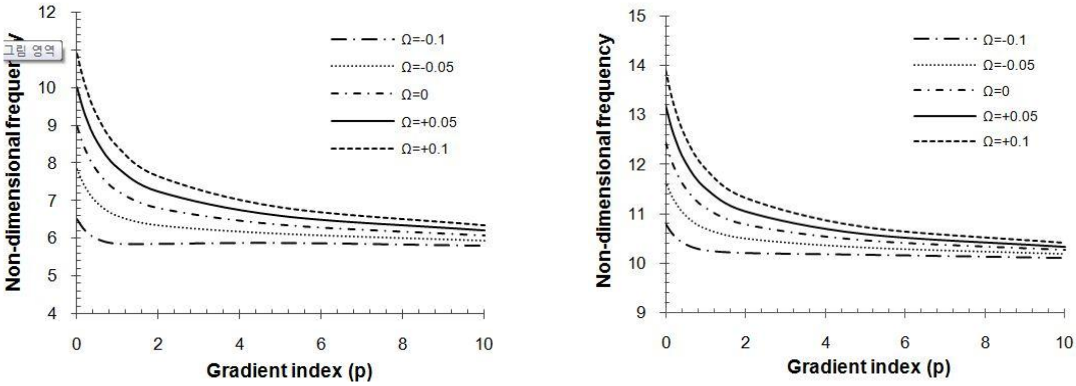
Table 6 Variation of the third dimensionless frequency of embedded S-S MEE-FG nanobeam for various nonlocal parameter, magnetic potentials and electric voltages (L/h = 20)

(K_w, K_p)	μ		Ω = -0.05			Ω = 0			Ω = +0.05		
			p = 0.2	p = 1	p = 5	p = 0.2	p = 1	p = 5	p = 0.2	p = 1	p = 5
(0,0)	0	V = -5	78.0770	71.9628	68.0821	78.9678	72.5258	68.2752	79.8487	73.0845	68.4677
		V = 0	77.9410	71.5329	67.3452	78.8334	72.0993	67.5404	79.7158	72.6613	67.7350
		V = +5	77.8048	71.1005	66.6001	78.6987	71.6703	66.7974	79.5826	72.2356	66.9942
	1	V = -5	56.3253	52.2809	49.8928	57.5538	53.0532	50.156	58.7566	53.8144	50.4177
		V = 0	56.1367	51.6876	48.8824	57.3692	52.4686	49.151	58.5758	53.2382	49.4180
		V = +5	55.9474	51.0874	47.8507	57.1840	51.8775	48.125	58.3944	52.6557	48.3977
	2	V = -5	46.0393	43.0414	41.4298	47.5344	43.9763	41.7463	48.9838	44.8917	42.0604
		V = 0	45.8083	42.3188	40.2073	47.3107	43.2693	40.5333	48.7668	44.1994	40.8568
		V = +5	45.5761	41.5837	38.9464	47.0859	42.5506	39.2829	48.5488	43.4960	39.6166
(25,5)	0	V = -5	80.9396	74.9652	71.1579	81.7992	75.5059	71.3427	82.6499	76.0427	71.5269
		V = 0	80.8084	74.5527	70.4532	81.6694	75.0963	70.6397	82.5215	75.6360	70.8258
		V = +5	80.6770	74.1379	69.7413	81.5394	74.6845	69.9297	82.3928	75.2272	70.1177
	1	V = -5	60.2307	56.3421	54.0145	61.3810	57.0595	54.2576	62.5102	57.7679	54.4997
		V = 0	60.0543	55.7920	53.0826	61.2080	56.5164	53.3300	62.3403	57.2315	53.5762
		V = +5	59.8774	55.2365	52.1341	61.0344	55.9680	52.3859	62.1699	56.6901	52.6366
	2	V = -5	50.7425	47.8927	46.3109	52.1028	48.7346	46.5942	53.4285	49.5621	46.8759
		V = 0	50.5331	47.2443	45.2205	51.8988	48.0976	45.5107	53.2296	48.9359	45.7990
		V = +5	50.3227	46.5870	44.1032	51.6940	47.4520	44.4007	53.0299	48.3016	44.6961

provide higher frequencies than positive voltages. Unlike the electric voltage, negative magnetic potentials provide lower natural frequencies than positive magnetic potentials. This is because tensile and compressive axial forces are generated in the nanobeam by applying negative and positive electric potentials, respectively.

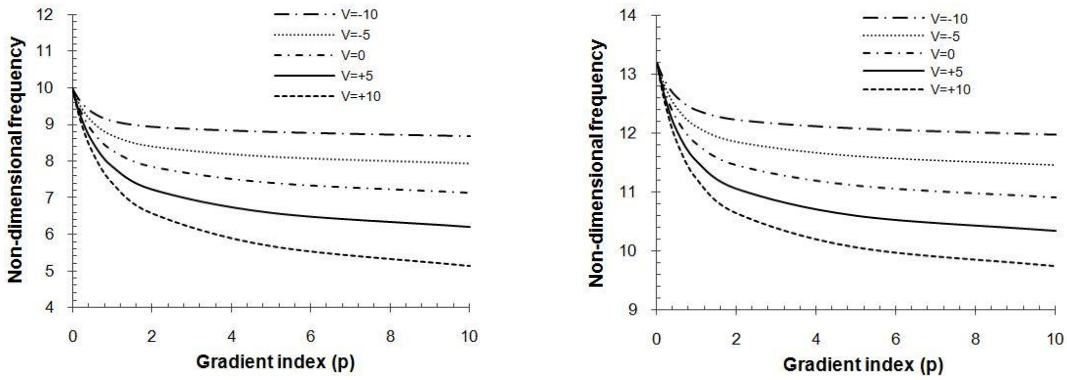
The effects of magnetic potential and electric voltage on the variations of the first non-dimensional frequency of the simply supported MEE-FG nanobeams versus power-law exponent with and without elastic foundation at L/h = 20 are presented in Figs. 2 and 3, respectively. It is observed that the non-dimensional frequency reduces with the increase of gradient index, especially for lower values of gradient index. Increasing gradient index leads to reduction in portion of ceramic phase and increment of metallic phase. So, increasing the portion of metallic phase reduces the stiffness of nanobeams and vibration frequencies decrease. But, this reduction is more considerable according to the positive values of magnetic potential and external electric voltage. Also, it is found that effect of higher values of gradient index on the magnetic potential is less than lower gradient indexes. Contrary to magnetic potential, the influence of higher gradient indexes on electric voltage is more than lower ones. In fact, by increasing gradient index the differences between frequency results of various electric voltages rise.

The effect of nonlocal parameter on the first frequency of the MEE-FG nanobeams versus power-law index is depicted in Fig. 4 (L/h = 20, V = +5, Ω = +0.1, K_w = 25, K_p = 5). It is



(a) $K_w = K_p = 0$ (b) $K_w = 25$, $K_p = 5$

Fig. 2 Effect of external magnetic potential on the dimensionless frequency of the S-S nanobeam with respect to gradient index (L/h = 20, V = +5, μ = 2)



(a) $K_w = K_p = 0$ (b) $K_w = 25$, $K_p = 5$

Fig. 3 Effect of external electric voltage on the dimensionless frequency of the S-S nanobeam with respect to gradient index (L/h = 20, Ω = +0.05, μ = 2)

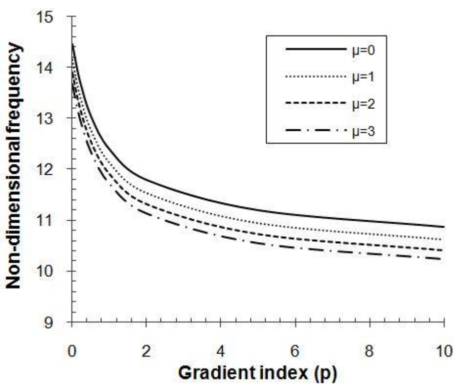
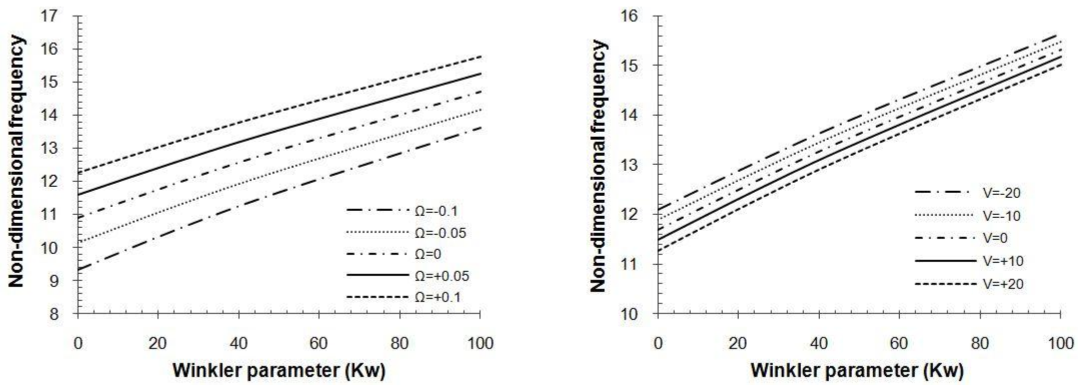


Fig. 4 Effect of nonlocal parameter on the dimensionless frequency of the S-S nanobeam with respect to gradient index (L/h = 20, V = +5, Ω = +0.1, K_w = 25, K_p = 5)

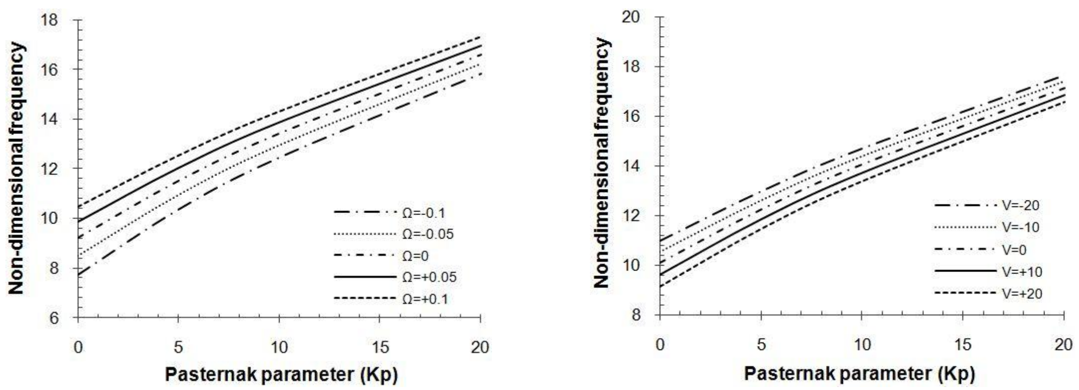
observable that both gradient index and nonlocal parameter have a softening influence on the beam structure and reduces the natural frequencies. Hence, nonlocal beam model provides lower frequencies compared to local beam model. So, with the rise of gradient index the non-dimensional frequency diminishes for all values of nonlocal parameter. The variations of the first fundamental frequency of MEE-FG nanobeams versus the Winkler and Pasternak parameters for various magnetic potentials and electric voltages at L/h = 20 are plotted in Figs. 5 and 6, respectively. According to these figures, regardless of the sign and magnitude of magnetic potential and electric voltage, the non-dimensional natural frequency increases with the rise of Winkler and Pasternak parameters, due to the increment in rigidity of the FGMEE nanobeam. Another observation is that effect of Pasternak elastic parameter on the non-dimensional frequency is more than Winkler parameter.

Hence, the shear layer of elastic foundation or Pasternak foundation has a significant influence on the frequency results. Figs. 7-8 illustrate the variations the dimensionless frequency of nonlocal



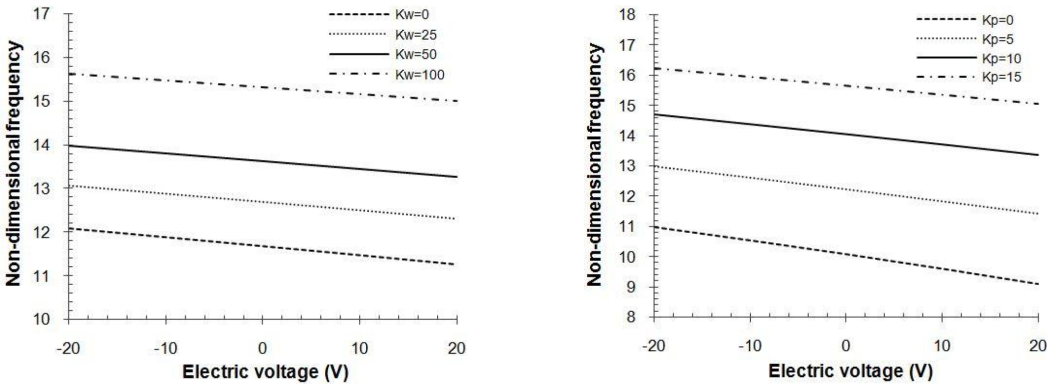
(a) (b)

Fig. 5 Effect of external magnetic potential and electric voltage on the dimensionless frequency of the S-S nanobeam with respect to Winkler parameter; (a) V = +5; (b) Ω = +0.05 (L/h = 20, μ = 2, p = 0.2)



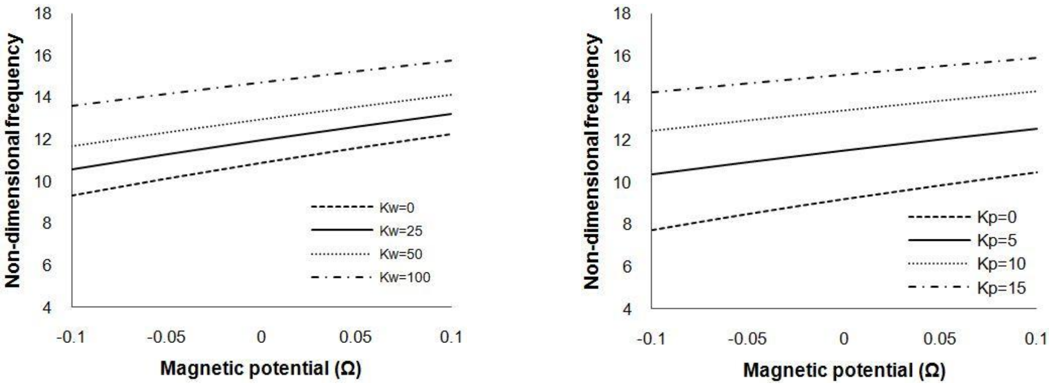
(a) (b)

Fig. 6 Effect of external magnetic potential and electric voltage on the dimensionless frequency of the S-S nanobeam with respect to Pasternak parameter; (a) V = +5, (b) Ω = +0.05 (L/h = 20, μ = 2, p = 0.2)



(a) $K_p = 5$, $p = 0.5$ (b) $K_w = 25$, $p = 0.5$

Fig. 7 Effect of elastic foundation on the dimensionless frequency of the S-S nanobeam with respect to electric voltage ($L/h = 20$, $\Omega = +0.05$, $\mu = 2$)

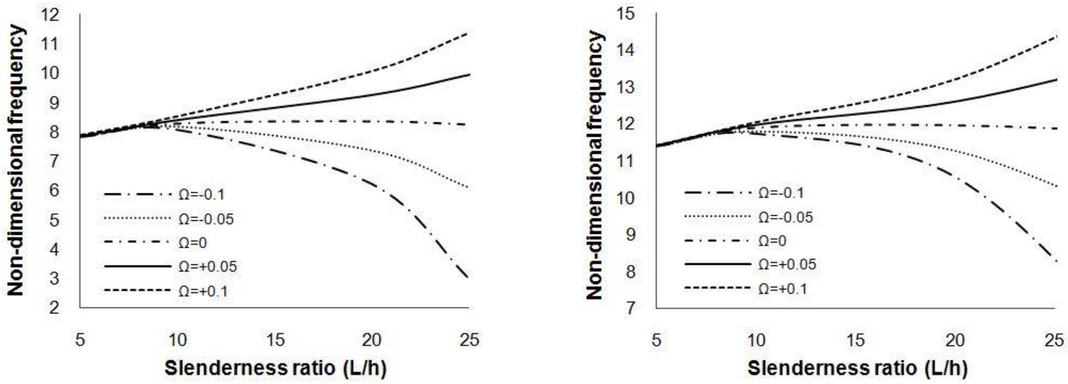


(a) $K_p = 5$, $p = 0.5$ (b) $K_w = 25$, $p = 0.5$

Fig. 8 Effect of elastic foundation on the dimensionless frequency of the S-S nanobeam with respect to magnetic potential ($L/h = 20$, $V = +5$, $\mu = 2$)

MEE-FG beams with respect to external voltage and magnetic potential at $L/h = 20$ for various Winkler and Pasternak constants. It is seen that external electric voltage has a reducing influence on the natural frequencies of MEE-FG nanobeams when it varies from negative values to positive one. But, it is found that the effect of magnetic potential is contrary to the electric voltage. So, when the magnetic potentials values changes from negative values to positive one, the non-dimensional frequency increases. Also, it is clearly observable that the influence of magnetic field on the frequency results is more than electric field.

Fig. 9 presents the variations of the non-dimensional natural frequency of embedded MEE-FG nanobeam with respect to slenderness ratio for different magnetic potentials and power-law index $p = 0.2$ and nonlocal parameter $\mu = 2$ $(mn)^2$. It is found that slenderness ratio has a significant influence on the frequency results of MEE-FG nanobeams. Thus, the dimensionless frequency is more affected according to the higher values of slenderness ratio. Also, it is seen that, positive values of magnetic potential has an increasing influence on natural frequencies of MEE-FG



(a) $K_w = K_p = 0$ (b) $K_w = 25$, $p = 0.5$

Fig. 9 Effect of slenderness ratio on the dimensionless frequency of the S-S nanobeam for various magnetic potentials ($L/h = 20$, $V = +5$, $\mu = 2$, $p = 0.2$)

nanobeams, while the negative values of magnetic potential show a reducing effect. In addition, it is clearly observable that the dimensionless frequency is approximately independent of slenderness ratio for zero magnetic potential ($\Omega = 0$).

5. Conclusions

This paper studies free vibration of MEE-FG nanobeams embedded in elastic foundation based on nonlocal higher order beam theory. Adopting Eringen's nonlocal elasticity theory to capture the small size effects, the nonlocal governing equations are derived and solved using analytical method. Magneto-electro-mechanical properties of the FG nanobeams are supposed to be position dependent based on power-law model. Provided numerical examples show the influences of elastic foundation parameters, magnetic potential, external electric voltage, gradient index, nonlocal parameter and slenderness ratio on the natural frequencies of MEE-FG nanobeams. It is observed that the nonlocal parameter and gradient index yields in reduction in both rigidity of the beam and natural frequencies. Also, the rigidity of the nonlocal MEE-FG beams and the frequency results increase with the rise of Winkler or Pasternak foundation parameters. Also, the fundamental frequencies depend on the sign and magnitude of the magnetic potential and electric voltage.

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Appendix

$$N-\mu\frac{\partial^2 N}{\partial x^2}=A_{xx}\frac{\partial u}{\partial x}+B_{xx}\frac{\partial \psi}{\partial x}-\alpha E_{xx}(\frac{\partial \psi}{\partial x}+\frac{\partial^2 w}{\partial x^2})+A_{31}^{e}\phi+A_{31}^{m}\gamma-N_E-N_H \tag{A1}$$

$$M-\mu\frac{\partial^2 M}{\partial x^2}=B_{xx}\frac{\partial u}{\partial x}+D_{xx}\frac{\partial \psi}{\partial x}-\alpha F_{xx}(\frac{\partial \psi}{\partial x}+\frac{\partial^2 w}{\partial x^2})+E_{31}^{e}\phi+E_{31}^{m}\gamma \tag{A2}$$

$$P-\mu\frac{\partial^2 P}{\partial x^2}=E_{xx}\frac{\partial u}{\partial x}+F_{xx}\frac{\partial \psi}{\partial x}-\alpha H_{xx}(\frac{\partial \psi}{\partial x}+\frac{\partial^2 w}{\partial x^2})+F_{31}^{e}\phi+F_{31}^{m}\gamma \tag{A3}$$

$$Q-\mu\frac{\partial^2 Q}{\partial x^2}=(A_{xz}-\beta D_{xz})(\frac{\partial w}{\partial x}+\psi)-E_{15}^{e}\frac{\partial \phi}{\partial x}-E_{15}^{m}\frac{\partial \gamma}{\partial x} \tag{A4}$$

$$R-\mu\frac{\partial^2 R}{\partial x^2}=(D_{xz}-\beta F_{xz})(\frac{\partial w}{\partial x}+\psi)-F_{15}^{m}\frac{\partial \phi}{\partial x}-F_{15}^{m}\frac{\partial \gamma}{\partial x} \tag{A5}$$

$$\int_{-h/2}^{h/2}\left\{D_x-\mu\frac{\partial^2 D_x}{\partial x^2}\right\}\cos(\xi z)dz=(E_{15}^{e}-\beta F_{15}^{e})(\frac{\partial w}{\partial x}+\psi)+F_{11}^{e}\frac{\partial \phi}{\partial x}+F_{11}^{m}\frac{\partial \gamma}{\partial x} \tag{A6}$$

$$\int_{-h/2}^{h/2}\left\{D_z-\mu\frac{\partial^2 D_z}{\partial x^2}\right\}\xi\sin(\xi z)dz=A_{31}^{e}\frac{\partial u}{\partial x}+(E_{31}^{e}-\alpha F_{31}^{e})\frac{\partial \psi}{\partial x}-\alpha F_{31}^{e}\frac{\partial^2 w}{\partial x^2}-F_{33}^{e}\phi-F_{33}^{m}\gamma \tag{A7}$$

$$\int_{-h/2}^{h/2}\left\{B_x-\mu\frac{\partial^2 B_x}{\partial x^2}\right\}\cos(\xi z)dz=(E_{15}^{m}-\beta F_{15}^{m})(\frac{\partial w}{\partial x}+\psi)+F_{11}^{m}\frac{\partial \phi}{\partial x}+X_{11}^{m}\frac{\partial \gamma}{\partial x} \tag{A8}$$

$$\int_{-h/2}^{h/2}\left\{B_z-\mu\frac{\partial^2 B_z}{\partial x^2}\right\}\xi\sin(\xi z)dz=A_{31}^{m}\frac{\partial u}{\partial x}+(E_{31}^{m}-\alpha F_{31}^{m})\frac{\partial \psi}{\partial x}-\alpha F_{31}^{m}\frac{\partial^2 w}{\partial x^2}-F_{33}^{m}\phi-X_{33}^{m}\gamma \tag{A9}$$

where $\mu = (e_0 a)^2$ and quantities used in above equations are defined as

$$\left\{A_{xx},B_{xx},D_{xx},E_{xx},F_{xx},H_{xx}\right\}=\int_{-h/2}^{h/2}c_{11}\left\{1,z,z^2,z^3,z^4,z^6\right\}dz \tag{A10}$$

$$\left\{A_{xz},D_{xz},F_{xz}\right\}=\int_{-h/2}^{h/2}c_{55}\left\{1,z^2,z^4\right\}dz \tag{A11}$$

$$\left\{A_{31}^{e},E_{31}^{e},F_{31}^{e}\right\}=\int_{-h/2}^{h/2}e_{31}\left\{\xi\sin(\xi z),z\xi\sin(\xi z),z^3\xi\sin(\xi z)\right\}dz \tag{A12}$$

$$\left\{E_{15}^{e},F_{15}^{e}\right\}=\int_{-h/2}^{h/2}e_{15}\left\{\cos(\xi z),z^2\cos(\xi z)\right\}dz \tag{A13}$$

$$\left\{F_{11}^{e}, F_{33}^{e}\right\} = \int_{-h/2}^{h/2} \left\{s_{11}\cos^2(\xi z), s_{33}\,\xi^2\sin^2(\xi z)\right\} dz \tag{A14}$$

$$\left\{A_{31}^{m}, E_{31}^{m}, F_{31}^{m}\right\} = \int_{-h/2}^{h/2} q_{31}\left\{\xi\sin(\xi z), z\xi\sin(\xi z), z^3\xi\sin(\xi z)\right\} dz \tag{A15}$$

$$\left\{E_{15}^{m}, F_{15}^{m}\right\} = \int_{-h/2}^{h/2} q_{15}\left\{\cos(\xi z), z^2\cos(\xi z)\right\} dz \tag{A16}$$

$$\left\{F_{11}^{m}, F_{33}^{m}\right\} = \int_{-h/2}^{h/2} \left\{d_{11}\cos^2(\xi z), d_{33}\,\xi^2\sin^2(\xi z)\right\} dz \tag{A17}$$

$$\left\{X_{11}^{m}, X_{33}^{m}\right\} = \int_{-h/2}^{h/2} \left\{\chi_{11}\cos^2(\xi z), \chi_{33}\,\xi^2\sin^2(\xi z)\right\} dz \tag{A18}$$

The explicit relation of the nonlocal normal force can be derived by substituting for the second derivative of N from Eq. (A1) into Eq. (28) as follows

$$\begin{aligned} N_x = {} & A_{xx}\frac{\partial u}{\partial x} + K_{xx}\frac{\partial \psi}{\partial x} - \alpha E_{xx}\frac{\partial^2 w}{\partial x^2} + A_{31}^{e}\phi + A_{31}^{m}\gamma - N_E - N_H + \mu(-\frac{\partial f}{\partial x} + I_0\frac{\partial^3 u}{\partial x \partial t^2} \\ & + \hat{I}_1\frac{\partial^3 \psi}{\partial x \partial t^2} - \alpha I_3\frac{\partial^4 w}{\partial x^2 \partial t^2}) \end{aligned} \tag{A19}$$

Also the explicit relation of the nonlocal bending moment can be derived by substituting for the second derivative of $\hat{M}$ from Eq. (A2) into Eq. (29) as follows

$$\begin{aligned} \hat{M} = {} & K_{xx}\frac{\partial u}{\partial x} + I_{xx}\frac{\partial \psi}{\partial x} - \alpha J_{xx}(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}) + (E_{31}^{e} - \alpha F_{31}^{e})\phi + (E_{31}^{m} - \alpha F_{31}^{m})\gamma \\ & + \mu(-\alpha\frac{\partial^2 P}{\partial x^2} - q + \frac{\partial}{\partial x}(N_E + N_H)\frac{\partial w}{\partial x}) + I_0\frac{\partial^2 w}{\partial t^2} + I_1\frac{\partial^3 u}{\partial x \partial t^2} + I_2\frac{\partial^3 \psi}{\partial x \partial t^2} \\ & - \alpha I_4(\frac{\partial^3 \psi}{\partial x \partial t^2} + \frac{\partial^4 w}{\partial x^2 \partial t^2}) + k_w w - k_p\frac{\partial^2 w}{\partial x^2}) \end{aligned} \tag{A20}$$

where

$$K_{xx} = B_{\mathrm{xx}} - \alpha E_{xx},\ I_{xx} = D_{\mathrm{xx}} - \alpha F_{xx},\ J_{xx} = F_{\mathrm{xx}} - \alpha H_{xx} \tag{A21}$$

By substituting for the second derivative of $\hat{Q}$ from Eq. (A4) into Eq. (30) the following expression for the nonlocal shear force is derived

$$\hat{Q} = \overline{A}_{xz}(\frac{\partial w}{\partial x} + \psi) - (E_{15}^{e} - \beta F_{15}^{e})\frac{\partial \phi}{\partial x} + \mu((N_H + N_E)\frac{\partial^3 w}{\partial x^3} - \alpha\frac{\partial^3 P}{\partial x^3} - \frac{\partial q}{\partial x} + k_w\frac{\partial w}{\partial x} - k_p\frac{\partial^3 w}{\partial x^3}$$
$$+ I_0\frac{\partial^3 w}{\partial x \partial t^2} + \alpha I_3\frac{\partial^4 u}{\partial x^2 \partial t^2} + \alpha I_4\frac{\partial^4 \psi}{\partial x^2 \partial t^2} - \alpha^2 I_6(\frac{\partial^5 w}{\partial x^3 \partial t^2} + \frac{\partial^4 \psi}{\partial x^2 \partial t^2}) - (E_{15}^{m} - \beta F_{15}^{m})\frac{\partial \gamma}{\partial x}) \quad \text{(A22)}$$

where

$$\overline{A}_{xz} = A^*_{xz} - \beta I^*_{xz},\; A^*_{xz} = A_{xz} - \beta D_{xz},\; I^*_{xz} = D_{xz} - \beta F_{xz} \quad \text{(A23)}$$