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A new refined nonlocal beam theory accounting for effect of thickness stretching in nanoscale beams

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Abstract. In this paper, a simple and refined nonlocal hyperbolic higher-order beam theory is proposed for bending and vibration response of nanoscale beams. The present formulation incorporates the nonlocal scale parameter which can capture the small scale effect, and it considers both shear deformation and thickness stretching effects by a hyperbolic variation of all displacements across the thickness without employing shear correction factor. The highlight of this formulation is that, in addition to modeling the displacement field with only two unknowns, the thickness stretching effect ($\varepsilon_z \neq 0$) is also included in the present model. By utilizing the Hamilton's principle and the nonlocal differential constitutive relations of Eringen, the equations of motion of the nanoscale beam are reformulated. Verification studies demonstrate that the developed theory is not only more accurate than the refined nonlocal beam theory, but also comparable with the higher-order shear deformation theories which contain more number of unknowns. The theoretical formulation proposed herein may serve as a reference for nonlocal theories as applied to the static and dynamic responses of complex-nanobeam-system such as complex carbon nanotube system.

Keywords: nonlocal theory; stretching effect; nanobeam

1. Introduction

Recent experimental results have demonstrated a significant size influence in mechanical characteristics when the dimensions of the structure become small. The local continuum models lack the capability of capturing such effects since they do not incorporate any internal length scale. Thus, these models are expected to fail when the structure size becomes comparable with the internal length scale(s) of the material. This motivated many authors to propose beam/plate theories based on size-dependent continuum models which consider the small scale influences. The nonlocal elasticity theory developed by Eringen (1972, 1983) is one of the promising size-dependent continuum models. Contrary to the local continuum models which suppose that the stress at a point is a function of strain at that point, the non-classical elasticity theory considers that

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the stress at a point is a function of strains at all points in the continuum. Thus, the small scale parameter is introduced through the employment of constitutive equations.

Based on the nonlocal elasticity model, a number of article have been published in the recent last years, attempting to propose nonlocal beam/plate models and use them to investigate the bending (Duan and Wang 2007, Lu, Zhang *et al.* 2007, Reddy and Pang 2008, Aghababaei and Reddy 2009, Reddy 2010, Berrabah, Tounsi *et al.* 2013, Tounsi, Benguediab *et al.* 2013a, Larbi Chaht, Kaci *et al*. 2015), buckling (Pradhan 2009, Murmu and Pradhan 2009a, Pradhan and Murmu 2009, Pradhan and Phadikar 2010, Amara, Tounsi *et al.* 2010, Tounsi, Semmah *et al.* 2013b, Tounsi, Benguediab *et al.* 2013c, Benguediab, Tounsi *et al.* 2014, Larbi Chaht, Kaci *et al*. 2015, Semmah, Tounsi *et al.* 2015), and vibration (Pradhan and Phadikar 2009ab; Murmu and Pradhan 2009b,c,d, Wang, Murmu *et al.* 2011, Pradhan and Kumar 2011, Pradhan and Sahu 2010, Zemri, Houari *et al.* 2015, Belkorissat, Houari *et al.* 2015, Chemi, Heireche *et al.* 2015) behaviors of nanoplates/nanobeams.

In recent years, researchers proposed some shear deformation theories to study bending, buckling and vibration behaviors of structures (Bellifa, Benrahou *et al.* 2016, Tounsi, Houari *et al.* 2016, Bourada, Amara *et al.* 2016, Houari, Tounsi *et al.* 2016, Ait Yahia, Ait Atmane *et al.* 2015, Ait Amar Meziane, Abdelaziz *et al.* 2014, Zidi, Tounsi *et al.* 2014, Bouderba, Houari *et al.* 2013, Tounsi, Houari *et al.* 2013d). In addition, the stretching thickness effect was studied by several authors to show its importance on mechanical behavior of structures (Bennoun, Houari *et al.* 2016, Bourada, Kaci *et al.* 2015, Hamidi, Houari *et al.* 2015, Belabed, Houari *et al.* 2014, Hebali, Tounsi *et al.* 2014). Recently, many papers have been published concerning with analysis of nanostructures. Among them, Ahouel, Houari *et al*. (2016) examined size-dependent mechanical response of functionally graded trigonometric shear deformable nanobeams including neutral surface position concept. Ebrahimi and Barati (2016) presented an exact solution for buckling analysis of embedded piezoelectromagnetically actuated nanoscale beams. Bounouara, Benrahou *et al*. (2016) developed a nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation. Eltaher, Khater *et al*. (2016) investigated the static stability of nonlocal nanobeams using higher-order beam theories.

In the present work, an analytical solution to the bending and vibration analyses of nanoscale beams is presented by proposing a novel nonlocal shear and normal deformation beam theory, which is compared with the predictions of other theories available in the literature. Just two unknown displacement functions are employed in the present model against four or more unknown displacement functions utilized in the corresponding ones. The effects due to small scale, transverse shear and thickness stretching are all included. The small scale influence is considered by utilizing the nonlocal constitutive relations of Eringen, while the shear and normal deformations effects are captured using the hyperbolic shear deformation theory (Zenkour 2013, Bourada *et al.* 2015). Based on the nonlocal constitutive relations of Eringen, equations of motion of nanoscale beams are obtained by employing Hamilton’s principle. Analytical solutions for deflection and natural frequency are presented for simply supported nanoscale beams, and the obtained results are compared with the existing solutions to check the accuracy of the present formulation.

2. Nonlocal beam model with thickness stretching effect

2.1 Kinematics

The displacement field of the hyperbolic shear deformation theory is proposed based on the supposition that the transverse shear stress vanishes on the top and bottom surfaces of the beam and is nonzero elsewhere. The displacement field is considered as (Bourada, Kaci *et al.* 2015)

$$\begin{aligned} u(x,z,t) &= -z\frac{\partial w_0}{\partial x} + f(z)\frac{\partial \varphi}{\partial x} \\ v(x,z,t) &= 0 \\ w(x,z,t) &= w_0(x,,t) + g(z)\,\varphi(x,t) \end{aligned} \tag{1}$$

where, w_0 is the displacement of the middle surface along the axis z; and the additional displacement φ accounts for the effect of normal stress (thickness stretching effect). In this work, the shape functions $f(z)$ and $g(z)$ are taken based on the hyperbolic function proposed by Zenkour (2013)

$$f(z) = h\sinh\left(\frac{z}{h}\right) - \left(\frac{4z^3}{3h^2}\right)\cosh\left(\frac{1}{2}\right), \text{ and } \quad g(z) = \frac{1}{12} f'(z) \tag{2}$$

The linear strain relations associated with the displacement field in Eq. (1) are

$$\varepsilon_x = z\,k_x + f(z)\,\eta_x \tag{3a}$$

$$\gamma_{xz} = \left[f'(z) + g(z)\right]\gamma_{xz}^0 \tag{3b}$$

$$\varepsilon_z = g'(z)\,\varepsilon_z^0 \tag{3c}$$

where the prime denotes differentiation with respect to z and, k_x, η_x, γ_{xz}^0, ε_z^0 are be defined

$$k_x = -\frac{\partial^2 w}{\partial x^2},\ \eta_x = \frac{\partial^2 \varphi}{\partial x^2},\ \gamma_{xz}^0 = \frac{\partial \varphi}{\partial x},\ \varepsilon_z^0 = \varphi \tag{4}$$

2.2 Equations of motion

Hamilton's principle is employed to determine the equations of motion

$$\int_{t_1}^{t_2} \left(\delta U + \delta V - \delta K\right) dt = 0 \tag{5}$$

where U, K and V represent the strain energy, kinetic energy and the work done by external forces, respectively.

The variation of the strain energy can be expressed as

$$\begin{aligned} \delta U &= \int_0^L \int_{-h/2}^{h/2} \left(\sigma_x \delta\varepsilon_x + \sigma_z \delta\varepsilon_z + \tau_{xz}\delta\gamma_{xz}\right) dz\,dx \\ &= \int_0^L \left(N\frac{d\delta u_0}{dx} - M\frac{d^2\delta w_0}{dx^2} - P\frac{d^2\delta\varphi}{dx^2} + N_z\delta\varphi + Q\frac{d\delta\varphi}{dx}\right) dx \end{aligned} \tag{6}$$

where M, P, N_z and Q are the stress resultants defined as

$$(M,P)=\int_{-h/2}^{h/2}(1,z,f(z))\,\sigma_x dz,\quad N_z=\int_{-h/2}^{h/2}\sigma_z g'(z)dz,\ \text{and}\ \ Q=\int_{-h/2}^{h/2}\tau_{xz}\left[f'(z)+g(z)\right]dz \tag{7}$$

The variation of work done by externally transverse loads q can be expressed as

$$\delta V=-\int_0^L q\delta\, w dx \tag{8}$$

The variation of the kinetic energy is obtained as

$$\begin{aligned}\delta K&=\int_0^L\int_{-h/2}^{h/2}\rho(z)\left[\dot{u}\delta\,\dot{u}+\dot{w}\delta\,\dot{w}\right]dzdx\\&=\int_0^L\Big\{I_0\dot{w}_0\delta\dot{w}_0+J_0\left[\dot{w}_0\delta\,\dot{\varphi}+\dot{\varphi}\,\delta\,\dot{w}_0\right]\\&+I_2\frac{d\dot{w}_0}{dx}\frac{d\delta\,\dot{w}_0}{dx}-J_2\left(\frac{d\dot{w}_0}{dx}\frac{d\delta}{dx}+\frac{d\dot{\varphi}}{dx}\frac{d\delta\,\dot{w}_0}{dx}\right)+K_0\dot{\varphi}\,\delta\,\dot{\varphi}+K_2\frac{d\dot{\varphi}}{dx}\frac{d\delta\,\dot{\varphi}}{dx}\Big\}dx\end{aligned} \tag{9}$$

Where dot-superscript convention indicates the differentiation with respect to the time variable t; and (I_i, J_i, K_i) are mass inertias defined as

$$\left(I_0,I_2\right)=\int_{-h/2}^{h/2}\left(1,z^2\right)\rho(z)dz \tag{10a}$$

$$\left(J_0,J_2\right)=\int_{-h/2}^{h/2}\left(g,zf\right)\rho(z)dz \tag{10b}$$

$$\left(K_0,K_2\right)=\int_{-h/2}^{h/2}\left(g^2,f^2\right)\rho(z)dz \tag{10c}$$

Substituting the expressions for δU, δV, and δK from Eqs. (6), (8), and (9) into Eq. (5) and integrating by parts, and collecting the coefficients of δw_0 and $\delta\varphi$, the following equations of motion of the beam are obtained

$$\delta\,w_0:\ \frac{d^2M}{dx^2}+q=I_0\ddot{w}_0+J_0\ddot{\varphi}-I_2\frac{d^2\ddot{w}_0}{dx^2}+J_2\frac{d^2\ddot{\varphi}}{dx^2} \tag{11a}$$

$$\delta\,\varphi:\ \frac{d^2P}{dx^2}-\frac{dQ}{dx}+N_z=-J_0\ddot{w}_0-J_2\frac{d^2\ddot{w}_0}{dx^2}-K_0\ddot{\varphi}+K_2\frac{d^2\ddot{\varphi}}{dx^2} \tag{11b}$$

2.3 Constitutive relations

The nonlocal theory considers that the stress at a point is related not only on the strain at that point but also on strains at all other points of the body. According to Eringen (1972, 1983), the nonlocal stress σ at a point is expressed as

$$\left(1-\mu\nabla^2\right)\sigma = t \tag{12}$$

where ∇^2 is the Laplacian operator, and t is the classical stress. $t=e_0a$ is the scale-effect parameter where e_0 is a material constant experimentally predicted, and a is an internal characteristic length (e.g., lattice parameter, molecular diameter, granular distance). For one dimensional beam element with considering thickness stretching effects, the nonlocal constitutive equation, Eq. (12), can be represented by

$$\left(1-\mu\frac{d^2}{dx^2}\right)\sigma_x = Q_{11}\varepsilon_x + Q_{13}\varepsilon_z \tag{13a}$$

$$\left(1-\mu\frac{d^2}{dx^2}\right)\tau_{xz} = Q_{55}\gamma_{xz} \tag{13b}$$

$$\left(1-\mu\frac{d^2}{dx^2}\right)\sigma_z = Q_{13}\varepsilon_x + Q_{33}\varepsilon_z \tag{13c}$$

Transforming the local stress resultants defined in Eq. (7), to nonlocal domain using the differential operator of Eringen, Eqs. (13), we obtain

$$\left(1-\mu\frac{d^2}{dx^2}\right)M = -D\frac{d^2w}{dx^2} + D_s\frac{d^2\varphi}{dx^2} + D_{st}\varphi \tag{14b}$$

$$\left(1-\mu\frac{d^2}{dx^2}\right)P = -D_s\frac{d^2w}{dx^2} + H_s\frac{d^2\varphi}{dx^2} + H_{st}\varphi \tag{14c}$$

$$\left(1-\mu\frac{d^2}{dx^2}\right)Q = A_s\frac{d\varphi}{dx} \tag{14d}$$

$$\left(1-\mu\frac{d^2}{dx^2}\right)N_z = -D_{st}\frac{d^2w}{dx^2} + H_{st}\frac{d^2\varphi}{dx^2} + F_{st}\varphi \tag{14f}$$

where

$$\left(D, D_s, H_s, F_{st}\right) = \int_A \left(z^2, z\,f, f^2, (g')^2\right)Q_{11}dA,\; A_s = \int_A g^2Q_{55}dA,\; \left(D_{st}, H_{st}\right) = \int_A g'\left(z, f\right)Q_{13}dA, \tag{15}$$

By substituting Eq. (14) into Eq. (11), the nonlocal equations of motion can be expressed in terms of displacements (w_0, φ) as

$$\begin{aligned} -D\frac{\partial^4 w_0}{\partial x^4} + D_s\frac{\partial^4\varphi}{\partial x^4} + D_{st}\frac{\partial^2\varphi}{\partial x^2} + q - \mu\frac{\partial^2 q}{\partial x^2} &= I_0\left(\ddot{w}_0 - \mu\frac{d^2\ddot{w}_0}{dx^2}\right) - I_2\left(\frac{d^2\ddot{w}_0}{dx^2} - \mu\frac{d^4\ddot{w}_0}{dx^4}\right) \\ &+ J_0\left(\ddot{\varphi} - \mu\frac{d^2\ddot{\varphi}}{dx^2}\right) + J_2\left(\frac{d^2\ddot{\varphi}}{dx^2} - \mu\frac{d^4\ddot{\varphi}}{dx^4}\right) \end{aligned} \tag{16a}$$

$$
\begin{aligned}
&-D_s \frac{\partial^4 w_0}{\partial x^4} - D_{st} \frac{\partial^2 w_0}{\partial x^2} + H_s \frac{\partial^4 \varphi}{\partial x^4} + 2H_{st} \frac{\partial^2 \varphi}{\partial x^2} - A_s \frac{\partial^2 \varphi}{\partial x^2} + Fst\varphi = -J_2\left(\frac{d^2 \ddot{w}_0}{dx^2} - \mu \frac{d^4 \ddot{w}_0}{dx^4}\right) \\
&-J_0\left(\ddot{w}_0 - \mu \frac{d^2 \ddot{w}_0}{dx^2}\right) + K_2\left(\frac{d^2 \ddot{\varphi}}{dx^2} - \mu \frac{d^4 \ddot{\varphi}}{dx^4}\right) - K_0\left(\ddot{\varphi} - \mu \frac{d^2 \ddot{\varphi}}{dx^2}\right)
\end{aligned} \tag{16b}
$$

3. Analytical solution of simply supported nanobeam

In this study, analytical solutions are given for simply supported isotropic nanobeams for bending and free vibration.

The following displacement field satisfies boundary conditions and governing equations.

$$
\begin{Bmatrix} w_0 \\ \varphi \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} W_n e^{i\omega t} \sin \beta x \\ \phi_n e^{i\omega t} \sin \beta x \end{Bmatrix} \tag{17}
$$

where W_n and ϕ_n are arbitrary parameters to be determined, ω is the eigenfrequency associated with n^{th} eigenmode, and $\beta = \frac{n\pi}{L}$. The transverse load q is also expanded in the Fourier sine series as

$$
q(x) = \sum_{n=1}^{\infty} Q_n \sin(\beta x),\ Q_n = \frac{2}{L}\int_0^L q(x)\sin(\beta x)dx \tag{18}
$$

The Fourier coefficients Q_n associated with some typical loads are given

$$
Q_n = q_0, \quad n = 1 \text{ for sinusoidal load,} \tag{19a}
$$

$$
Q_n = \frac{4q_0}{n\pi}, \quad n = 1,3,5 \ldots \text{for uniform load,} \tag{19b}
$$

$$
Q_n = \frac{2q_0}{L}\sin\frac{n\pi}{2}, \quad n = 1,2,3 \ldots \text{for point load } Q_0 \text{ at the midspan,} \tag{19c}
$$

Substituting the expansions of w_0, φ, ’ and q from Eqs. (17) and (18) into Eq. (16), the closed form solutions can be obtained from the following equations

$$
\left(\begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} - \lambda\omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}\right)\begin{bmatrix} W_n \\ \phi_n \end{bmatrix} = \lambda \begin{bmatrix} Q_n \\ 0 \end{bmatrix}, \tag{20}
$$

where

$$
\begin{aligned}
&S_{11} = D\beta^4, S_{12} = -D_s\beta^4 + D_{st}\beta^2, S_{22} = H_s\beta^4 - 2H_{st}\beta^2 + A_s\beta^2 + F_{st}, \lambda = 1 + \mu\beta^2 \\
&m_{11} = I_0 + I_2\beta^2, m_{12} = J_0 - J_2\beta^2, m_{22} = K_0 + K_2\beta^2.
\end{aligned} \tag{21}
$$

Table 1 Dimensionless maximum center deflection $\overline{w}$ under uniform load for simply supported nanobeams

L/h	μ (nm^2)	EBT	TBT	RBT	SBT (ε_z=0)	Tounsi, Benguediab *et al.* (2013a) ($\varepsilon_z\neq0$)	Present ($\varepsilon_z\neq0$)
5	0	1.3021	1.4321	1.4320	1.4317	1.4185	1.4138
	1	1.4271	1.5674	1.5673	1.5671	1.5529	1.5476
	2	1.5521	1.7028	1.7027	1.7025	1.6872	1.6814
	3	1.6771	1.8381	1.8381	1.8379	1.8215	1.8152
	4	1.8021	1.9734	1.9735	1.9733	1.9559	1.9490
10	0	1.3021	1.3346	1.3346	1.3345	1.3299	1.3270
	1	1.4271	1.4622	1.4622	1.4621	1.4571	1.4539
	2	1.5521	1.5898	1.5898	1.5897	1.5843	1.5809
	3	1.6771	1.7173	1.7174	1.7173	1.7115	1.7078
	4	1.8021	1.8449	1.8450	1.8449	1.8387	1.8347
20	0	1.3021	1.3102	1.3102	1.3102	1.3077	1.3077
	1	1.4271	1.4359	1.4359	1.4358	1.4331	1.4332
	2	1.5521	1.5615	1.5615	1.5615	1.5585	1.5586
	3	1.6771	1.6871	1.6872	1.6871	1.6839	1.6841
	4	1.8021	1.8128	1.8128	1.8128	1.8093	1.8095
100	0	1.3021	1.3024	1.3024	1.3024	1.3005	1.3023
	1	1.4271	1.4274	1.4274	1.4274	1.4254	1.4273
	2	1.5521	1.5525	1.5525	1.5525	1.5502	1.5523
	3	1.6771	1.6775	1.6775	1.6775	1.6751	1.6773
	4	1.8021	1.8025	1.8025	1.8025	1.7999	1.8024

4. Numerical results

Through this section, the effect of thickness stretching in nanobeam, nonlocality effect and slenderness ratios on the deflections and natural frequencies of the nanobeam will be discussed. The obtained results are compared with those predicted using the Euler-Bernoulli beam theory (EBT), Timoshenko beam theory (TBT), Reddy's beam theory (RBT) and the model of Berrabah, Tounsi *et al.* (2013) for a wide range of nonlocal parameter and slenderness ratio. For all computations, the shear correction factor and Poisson's ratio are considered as 5/6 and 0.3, respectively. The length of nanobeam L is supposed to be 10 nm. A conservative estimate of the nonlocal scale parameter $0\leq e_0a\leq2$ nm for single-walled carbon nanotubes (SWCNTs) is proposed by Wang (2005). Hence, in this work, the nonlocal parameter is taken as $\mu=(e_0a)^2$=0,1,2,3 and 4 nm to examine the nonlocal effects on the responses of nanobeam. For convenience, the following non-dimensional quantities are employed:

- $\overline{w}=100w\dfrac{EI}{q_0L^4}$ for uniform load;
- $\overline{\omega}=\omega L^2\sqrt{\dfrac{I_0}{EI}}$ frequency parameter;

Table 1 illustrates the variation of the non-dimensional maximum deflections $\overline{w}$ with respect to nonlocal scale parameter, proposed theories, and slenderness ratios. A simply supported nanobeam subjected to uniform load is considered in this example and the calculated values are obtained using 100 terms in series in Eqs. (17) and (18). The predicted results are compared to those given by the Euler-Bernoulli beam theory (EBT), Timoshenko beam theory (TBT), Reddy's beam theory (RBT), Sinusoidal beam theory (SBT) of Berrabah, Tounsi *et al*. (2013) and the theory developed by Tounsi, Benguediab *et al*. (2013a). For all theories, it is noted that the deflection increases as the nonlocal scale parameter increases at a specified slenderness ratio. Moreover, for high slenderness (L/h=100) ratio, all theories are approximately identical in predicting the deflection, which confirms the accuracy of the simple Euler-Bernoulli model in the case of thin nanoscale beams. However, the discrepancy between EBT and other theories is noticeable for a moderately thick beam (L/h=10). On the other hand, the results predicted by employing the TBT coincide with those obtained using higher-order theories suggesting the accuracy of utilizing TBT for the case of moderately thick beams. It can be seen that the results from TBT, RBT and SBT due to ignoring the thickness stretching effect (ε_z=0) are slightly overestimate when comparing with those from the present theory (quasi-3D, $\varepsilon_z \neq 0$). This effect is more pronounced on thick beams (L/h=5). Noted that the present model has only three unknowns as in the case of TBT, RBT and SBT, while the number of unknowns in quasi-3D (Tounsi, Benguediab *et al.* 2013a) is four. Also, the present theory does not required shear correction coefficients as in the case of TBT.

Table 2 Dimensionless fundamental frequency ϖ of simply supported nanobeam

L/h	μ (nm^2)	EBT	TBT	RBT	SBT (ε_z=0)	Tounsi, Benguediab *et al.* (2013a) ($\varepsilon_z \neq 0$)	Present ($\varepsilon_z \neq 0$)
5	0	9.7112	9.2740	9.2745	9.2752	9.2993	9.3211
	1	9.2647	8.8477	8.8482	8.8488	8.8718	8.8926
	2	8.8747	8.4752	8.4757	8.4763	8.4983	8.5182
	3	8.5301	8.1461	8.1466	8.1472	8.1683	8.1874
	4	8.2228	7.8526	7.8530	7.8536	7.8740	7.8925
10	0	9.8293	9.7075	9.7075	9.7077	9.7197	9.7307
	1	9.3774	9.2612	9.2612	9.2614	9.2728	9.2834
	2	8.9826	8.8713	8.8714	8.8715	8.8825	8.8926
	3	8.6338	8.5269	8.5269	8.5271	8.5376	8.5473
	4	8.3228	8.2196	8.2197	8.2198	8.2300	8.2393
20	0	9.8595	9.8281	9.8281	9.8282	9.8365	9.8358
	1	9.4062	9.3763	9.3763	9.3764	9.3843	9.3837
	2	9.0102	8.9816	8.9816	8.9816	8.9892	8.9886
	3	8.6604	8.6328	8.6328	8.6329	8.6402	8.6396
	4	8.3483	8.3218	8.3218	8.3218	8.3289	8.3283
100	0	9.8692	9.8679	9.8679	9.8679	9.8750	9.8749
	1	9.4155	9.4143	9.4143	9.4143	9.4211	9.4210
	2	9.0191	9.0180	9.0180	9.0180	9.0245	9.0244
	3	8.6689	8.6678	8.6678	8.6678	8.6740	8.6739
	4	8.3566	8.3555	8.3555	8.3555	8.3615	8.3614

Table 3 The first three dimensionless frequency $\bar{\varpi}$ of simply supported nanobeam (L/h=5)

Modes (n)	μ (nm^2)	EBT	TBT	RBT	SBT (ε_z=0)	Tounsi, Benguediab *et al.* (2013a) ($\varepsilon_z\neq0$)	Present ($\varepsilon_z\neq0$)
1	0	9.7112	9.2740	9.2745	9.2752	9.2993	9.3211
	1	9.2647	8.8477	8.8482	8.8488	8.8718	8.8926
	2	8.8747	8.4752	8.4757	8.4763	8.4983	8.5182
	3	8.5301	8.1461	8.1466	8.1472	8.1683	8.1874
	4	8.2228	7.8526	7.8530	7.8536	7.8740	7.8925
2	0	37.1120	32.1665	32.1847	32.1948	32.3947	32.4491
	1	31.4239	27.2364	27.2519	27.2604	27.4297	27.4757
	2	27.7422	24.0453	24.0589	24.0664	24.2159	24.2565
	3	25.1104	21.7642	21.7765	21.7833	21.9186	21.9554
	4	23.1088	20.0293	20.0407	20.0470	20.1714	20.2053
3	0	78.0234	61.4581	61.5746	61.6192	62.1977	62.1968
	1	56.7798	44.7247	44.8095	44.8420	45.2629	45.2623
	2	46.8246	36.8831	36.9531	36.9798	37.3270	37.3265
	3	40.7568	32.1036	32.1645	32.1878	32.4900	32.4895
	4	36.5657	28.8023	28.8569	28.8778	29.1489	29.1485

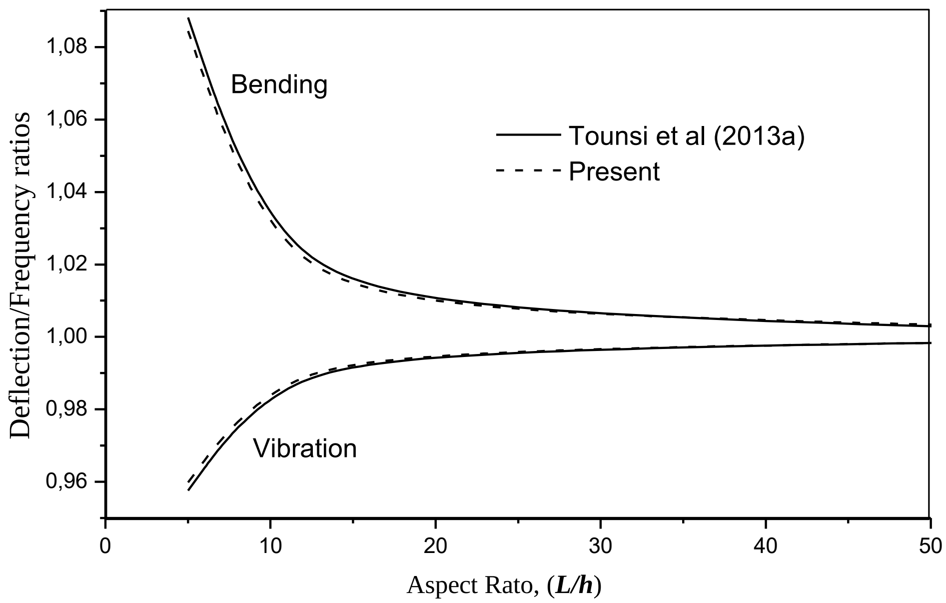

Fig. 1 Effect of the aspect ratio on the deflection, and fundamental frequency ratios for a simply supported nanobeam with e_0a=1 nm

The non-dimensional frequency $\bar{\varpi}$ of a simply supported nanoscale beam are shown in Tables 2 and 3 for various values of scale parameter μ and four different values of slenderness ratio (L/h=5, 10, 20, 100) based on analytical Navier solution technique. It can be concluded from these results that an increase in nonlocal parameter gives rise to a decrement in the frequency. In addition, it is seen that the $\overline{\varpi}$ increases by increasing slenderness ratio (*L/h*) and it can be stated that nonlocality factor has a notable influence on the frequency and especially at the higher vibration modes (see Table 3). Our results are in good agreement with those obtained by Berrabah, Tounsi *et al*. (2013 for EBT, TBT, RBT, and SBT. However, it can be seen that, the inclusion of thickness stretching effect (i.e., $\varepsilon_z \neq 0$) leads to a slight increase of frequency.

Fig. 1 demonstrates the variation of deflection and frequency ratios of nanobeam with the slenderness ratio (*L/h*). In this example, the deflection, and frequency ratios are defined as the ratios of those computed by present formulation to the correspondences computed by EBT where the shear deformation effect is neglected. Observing this figure, it is easily deduced that, the influence of slenderness ratio is to decrease the natural frequencies and increase the deflections, and this effect is considerable for thick beams at higher vibration modes (see Fig. 2). This demonstrates that the slenderness ratio effect results in a reduction of the beam stiffness. Also it can be concluded from the results of the Fig. 1 that the present nonlocal model is capable to produce very accurate results compared with the nonlocal theory developed by Tounsi, Benguediab *et al*. (2013a) with higher number of unknowns.

The influence of the nonlocal scale parameter on the bending and vibration behaviors of nanoscale beam is shown in Fig. 3. The transverse displacement, and frequency ratios are defined as the ratios of those calculated by the nonlocal theory to the correspondences calculated by the local theory (i.e., μ=0). This figure demonstrates a nonlinear variation of the bending and vibration responses with the nonlocal scale parameter. It can be observed that the transverse displacement ratio is greater than unity, whereas the frequency ratios are smaller than unity. It means that the local theory under-estimates the transverse displacements and over-estimates the frequencies of the nanoscale beams compared to the nonlocal one. This is due to the fact that the local model is

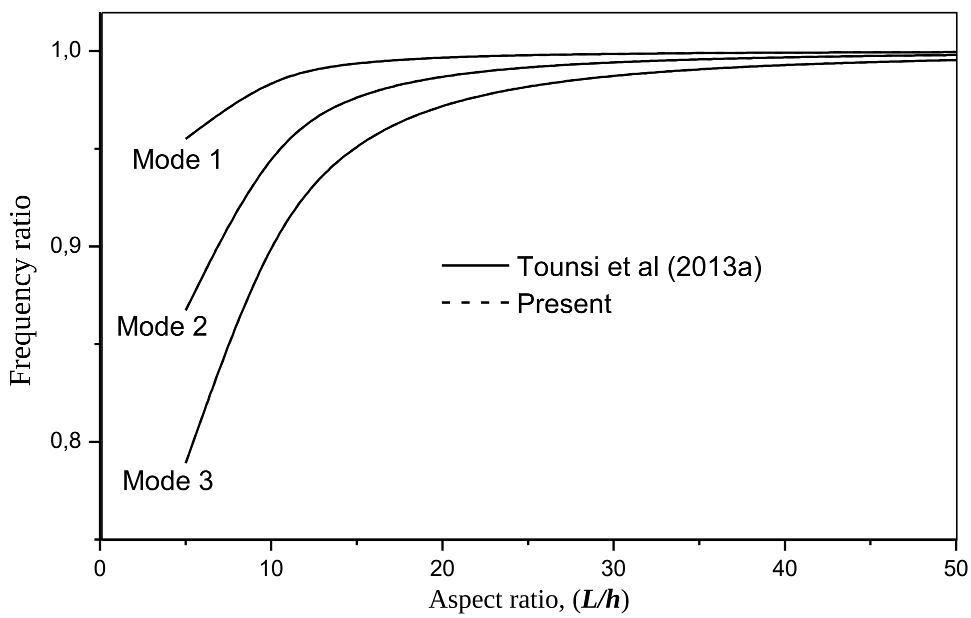

Fig. 2 Effect of the aspect ratio on higher frequency ratios for a simply supported nanobeam with e_0a=1 nm

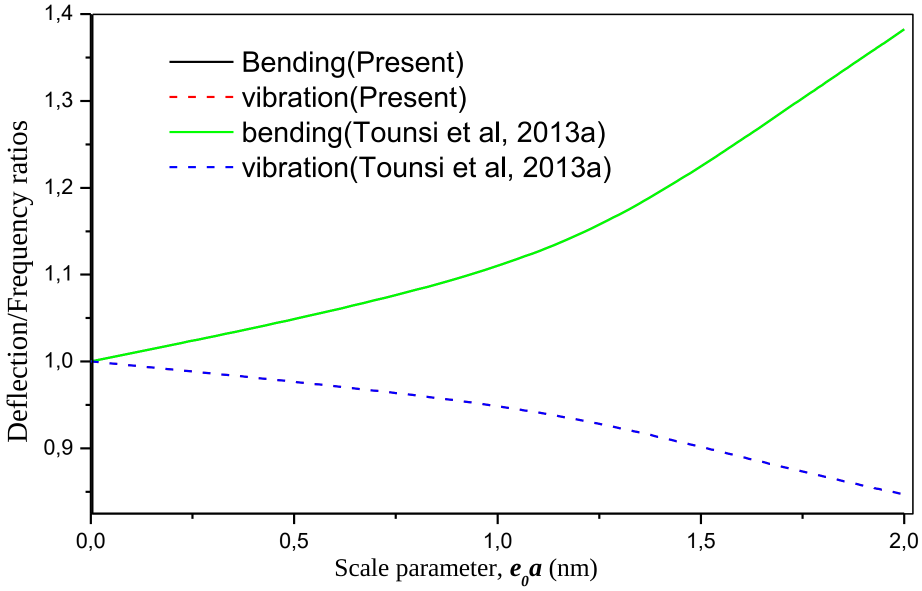

Fig. 3 Effect of the scale parameter on the defection, and fundamental frequency ratios for a simply supported nanobeam with *L/h*=10

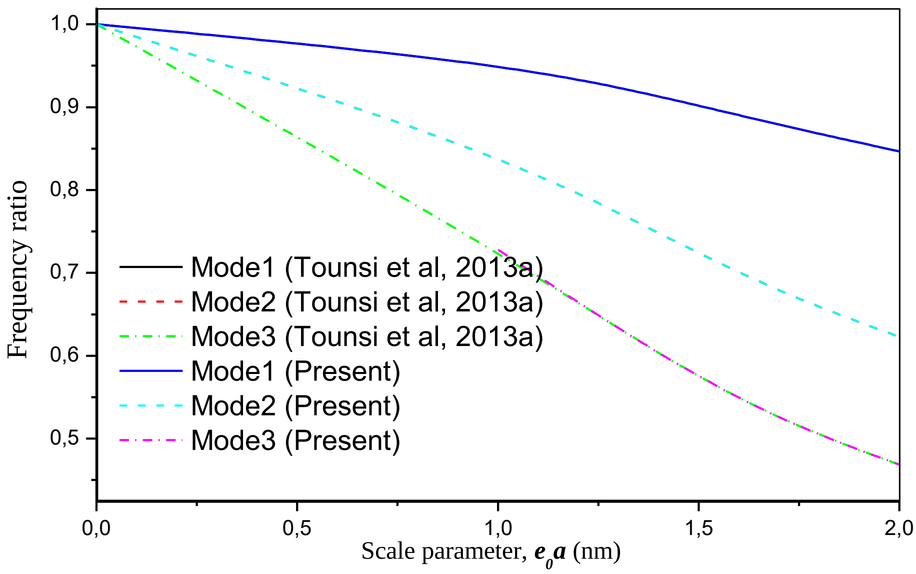

Fig. 4 Effect of the aspect ratio on higher frequency ratios for a simply supported nanobeam with e_0a=1 nm

unable to consider the small scale influence of the nanoscale beams. The difference between the local and nonlocal models is especially important for the higher modes (see Fig. 4).

5. Conclusions

A novel nonlocal thickness-stretching hyperbolic shear deformation beam theory is proposed for the bending, and dynamic behavior of nanobeams. The present theory is able to consider the small scale, shear deformation and thickness stretching influences of nanoscale beams, and respects the zero traction boundary conditions on the upper and lower surfaces of the nanoscale beam without employing shear correction coefficient. From Hamilton's principle as well as nonlocal elasticity theory of Eringen, the nonlocal equations of motion are obtained according to the refined two-variable shear deformation beam theory and then solved via an exact analytical solution. Results demonstrate that the incorporation of thickness stretching influence makes a nanoscale beam stiffer, and hence, leads to a diminishing of transverse displacement and an increase of frequency. However, it is remarked that the consideration of the nonlocal parameter and shear deformation influences lead to an increase in the transverse displacements and a reduction of the natural frequencies of nanoscale beams.

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