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On the vibration of aligned carbon nanotube reinforced composite beams

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Abstract. Carbon nanotubes have exceptional mechanical, thermal and electrical properties, and are considered for high performance structural and multifunctional composites. In the present study, the natural frequencies of aligned single walled carbon nanotube (CNT) reinforced composite beams are obtained using shear deformable composite beam theories. The Ritz method with algebraic polynomial displacement functions is used to solve the free vibration problem of composite beams. The Mori-Tanaka method is applied to find the composite beam mechanical properties. The continuity conditions are satisfied among the layers by modifying the displacement field. Results are found for different CNT diameters, length to thickness ratio of the composite beam and different boundary conditions. It is found that the use of smaller CNT diameter in the reinforcement element gives higher fundamental frequency for the composite beam.

Keywords: carbon nanotubes; nano composites; mechanical properties; vibration; laminate theory

1. Introduction

Due to their exceptional mechanical properties, carbon nanotubes (CNTs) can be used as reinforcement in nanocomposites (Treacy *et al*. 1996, Goze *et al*. 1999, Thostenson *et al*. 2001, Shi *et al*. 2004, Wuite and Adali 2005). Using CNT as reinforcement in polymer composites may provide an increase in stiffness and strength relative to the carbon fiber-polymer composites. One of the key steps in composite analysis is the determination of the constitutive relations that gives the bulk mechanical properties of composites. Several attempts have been made to derive the constitutive modeling of nanotube-reinforced polymer composites. Odegard *et al*. 2002 used an equivalent-continuum modeling method in order to determine the bulk material properties of Single Walled Carbon Nanotube (SWCNT) polymer composites with various nanotube lengths, concentrations and orientations. Shi *et al*. (2004) used the Mori-Tanaka effective-field method to calculate the effective elastic moduli of composites with aligned or randomly nanotubes.

The static and dynamic behavior of carbon fiber reinforced composites has been extensively studied in past researches (Aydogdu 2005, 2006a, 2006b, Leissa and Narita 1989, Baharlou and Leissa 1987). However, very few studies can be found on the static analysis of carbon nanotube reinforced composites. Wuite and Adali (2005) studied the deflection and the stress behavior of nanocomposite reinforced beams using a multiscale analysis. Maghamikia and Jam (2011) studied

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buckling analysis of circular and annular composite plate reinforced with carbon nanotubes using Finite Element Method. They used the Mori-Tanaka method in order to obtain stiffness coefficients. Nonlinear free vibration of functionally graded carbon nanotube-reinforced composite beams investigated by Ke *et al*. (2010). Stimulated by the concept of FGMs, CNT-based composite plates were proposed by Shen (2009); nonlinear bending analysis of functionally graded carbon nanotube-reinforced composite (FG-CNTRC) plates in thermal environments was studied. By using the finite element method (FEM), bending and free vibration analyses were carried out for various types of FG-CNTRC plates by Zhou *et al*. (2012). Free vibration of functionally graded carbon nanotube-reinforced composite plates using the element-free kp-Ritz method in thermal environment is studied by Lei *et al*. (2013). In real applications, it is difficult to alligne CNTs in a matrix. Agglomeration is another problem when using CNT as a reinforcement element. Due to this reasons experimental works are required in order to understand mechanical behavior of CNT reinforced composites. As a starting point theoretical works are also required.

The main objective of this paper is to investigate the vibration behavior of aligned carbon nanotube reinforced composite beams. The constitutive relations are obtained using the Mori Tanaka effective field method for the aligned CNT reinforced composites. Different shear deformation models are used in the analysis. The Ritz method is used to get the natural frequencies for the CNT reinforced composite beam with different boundary conditions.

2. Analysis

2.1 Micromechanics model

Consider a polymer composite reinforced with aligned SWCNTs that are straight and infinitely long. The matrix is assumed to be elastic and isotropic, with Young modulus E_m and Poisson's ratio ν_m. Each straight CNT has transversely isotropic elastic properties.

The composite material is also transversely isotropic with following stress-strain relations (Hill 1965)

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{Bmatrix}=\begin{bmatrix}n & l & l & 0 & 0 & 0\\ l & k+m & k-m & 0 & 0 & 0\\ l & k-m & k+m & 0 & 0 & 0\\ 0 & 0 & 0 & 2m & 0 & 0\\ 0 & 0 & 0 & 0 & 2p & 0\\ 0 & 0 & 0 & 0 & 0 & 2p\end{bmatrix}\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{23}\\ \varepsilon_{13}\\ \varepsilon_{12}\end{Bmatrix} \tag{1}$$

where k, l, m, n and p are the Hill's elastic moduli (Hill 1965), n is the uniaxial tension modulus in the fiber direction (direction 1), k is the plane strain bulk modulus normal to the fiber direction, l is the associated cross-modulus, m and p are the shear moduli in planes normal and parallel to the fiber direction, respectively (Shi *et al*. 2004). Using the Mori-Tanaka method, the Hill's elastic moduli are found as (Shi *et al*. 2004, Wuite and Adali 2005)

$$k=\frac{E_m\{E_m c_m+2k_r(1+\nu_m)[1+c_r(1-2\nu_m)]\}}{2(1+\nu_m)\left[E_m(1+c_r-2\nu_m)+2c_m k_r\left(1-\nu_m-2\nu_m^2\right)\right]}$$

$$l = \frac{E_m\left\{c_m\nu_m\left[E_m + 2k_r(1+\nu_m)\right] + 2c_r l_r\left(1-\nu_m^2\right)\right\}}{(1+\nu_m)\left[E_m(1+c_r-2\nu_m) + 2c_m k_r\left(1-\nu_m-2\nu_m^2\right)\right]}$$

$$n = \frac{E_m^2 c_m(1+c_r-c_m\nu_m)}{(1+\nu_m)\left[E_m(1+c_r-2\nu_m) + 2c_m k_r\left(1-\nu_m-2\nu_m^2\right)\right]} + \frac{E_m\left[2c_m^2 k_r(1-\nu_m) + c_r n_r(1-2\nu_m+c_r) - 4c_m l_r \nu_m\right]}{E_m(1+c_r-2\nu_m) + 2c_m k_r\left(1-\nu_m-2\nu_m^2\right)}$$

$$l = \frac{E_m\left[E_m c_m + 2(1+c_r)p_r(1+\nu_m)\right]}{2(1+\nu_m)\left[E_m(1+c_r) + 2c_m p_r(1+\nu_m)\right]}$$

$$m = \frac{E_m\left[E_m c_m + 2m_r(1+\nu_m)(3+c_r-4\nu_m)\right]}{2(1+\nu_m)\left\{E_m\left[c_m + 4c_r(1-\nu_m)\right] + 2c_m m_r\left(3-\nu_m-4\nu_m^2\right)\right\}} \tag{2}$$

where k_r, l_r, m_r, n_r and p_r are the Hill's elastic moduli for reinforcing phase and c_r and c_m are the volume fraction of reinforcing phase and matrix respectively. Material properties for a unidirectional CNT reinforced composite can be given as follows (Shi *et al*. 2004, Wuite and Adali 2005)

$$E_L = n - \frac{l^2}{k}, \qquad E_T = \frac{4m\left(kn-l^2\right)}{kn-l^2+mn}$$
$$G_{LT} = 2p \qquad \nu_{LT} = \frac{l}{2k} \tag{3}$$

Where subscripts L and T denote longitudinal and transverse directions of composite beam respectively.

2.2 Carbon nanotube reinforced composite beams

In this section, derivation of the governing equations for the laminated composite beams is explained. Consider a straight uniform composite beam of length a, height h and width b. The beam is assumed to be constructed of arbitrary number, N, of linearly elastic transversely isotropic layers. Therefore, the stress state in each layer is given by Aydogdu (2005)

$$\sigma_x^{(\chi)} = Q_{11}^{(\chi)}\varepsilon_x, \qquad \tau_{xz}^{(\chi)} = Q_{55}^{(\chi)}\gamma_{xz}, \tag{4}$$

where $Q_{ij}^{(\chi)}$ the well-known reduced stiffnesses (Herakovich 1998) and χ is the number of layers. Assuming that the deformations of the beam take place in the x-z plane and upon denoting the displacement components along the x, y and z directions by U, V and W respectively, the following displacement field can be written

$$\begin{aligned} U(x,z;t) &= u(x;t) - zw_{,x} + f(z)u_1(x;t), \\ V(x,z;t) &= 0, \\ W(x,z;t) &= w(x;t), \end{aligned} \tag{5}$$

Where u_1 is the shear deformation at mid-plane. The displacement model (5) yields the following kinematic relations

$$\varepsilon_x = u_{,x} - zw_{,xx} + f(z)u_{1,x}, \qquad \gamma_{xz} = f'u_1, \tag{6}$$

where a prime denotes the derivative with respect to z and "$_{,x}$" represent partial derivative with respect to x.

Although different shape functions are applicable, only the ones which convert the present theory to the corresponding parabolic shear deformation beam theory (PSDBT) of Reddy (1984), the first order shear deformation beam theory (FSDBT) of Mindlin (1951) and the general exponential shear deformation beam theory (GESDBT) of Aydogdu (2009) are employed in the present study. This is achieved by choosing the shape functions as follows

$$\begin{aligned} &FSDBT: \; f(z) = z, \\ &PSDBT: \; f(z) = z(1 - 4z^2/3h^2), \\ &GESDBT: \; f(z) = (z)(\alpha)^{\frac{-2(z/h)^2}{\ln\alpha}} \; . \end{aligned} \tag{7}$$

In Aydogdu (2009), it is shown that α=3 is best choice in the static and dynamic analysis of laminated composite plates and beams. By substituting the stress-strain relations into the expressions of the force and moment resultants of the present theory the following constitutive equations are obtained (Aydogdu 2005, 2006a, 2006b)

$$\begin{bmatrix} N_x^c \\ M_x^c \\ M_x^s \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} & E_{11} \\ & D_{11} & F_{11} \\ Sim & & H_{11} \end{bmatrix} \begin{bmatrix} u_{,x} \\ -w_{,xx} \\ u_{1,x} \end{bmatrix} \quad [Q_x^s] = [A_{55}][u_1] \tag{8}$$

The extensional, coupling, bending and transverse shear rigidities are defined as follows

$$\begin{aligned} &A_{11} = \int_{-h/2}^{h/2} Q_{11}^{(\chi)} dz, \quad A_{55} = \int_{-h/2}^{h/2} Q_{55}^{(\chi)} (f')^2 dz, B_{11} = \int_{-h/2}^{h/2} Q_{ij}^{(\chi)} z dz, \quad E_{11} = \int_{-h/2}^{h/2} Q_{ij}^{(\chi)} f(z) dz, \\ &D_{11} = \int_{-h/2}^{h/2} Q_{ij}^{(\chi)} z^2 dz, \quad F_{11} = \int_{-h/2}^{h/2} Q_{ij}^{(\chi)} f(z) z dz, \quad H_{11} = \int_{-h/2}^{h/2} Q_{ij}^{(\chi)} (f)^2 \; dz \\ &(\;)' = d(\;)/dz. \end{aligned} \tag{9}$$

The force and moment resultants are defined in the following form

$$\begin{aligned} &(N_x^c) = \int_{-h/2}^{h/2} (\sigma_x) dz, \quad (M_x^c) = \int_{-h/2}^{h/2} \sigma_x z \, dz, \\ &(M_x^s) = \int_{-h/2}^{h/2} (\sigma_x) f(z) dz, \quad (Q_x^s) = \int_{-h/2}^{h/2} (\tau_{xz}) f'(z) d\,z \end{aligned} \tag{10}$$

In these definitions, the resultants denoted with a superscript 'c' are the conventional ones of the classical beam theories; whereas the remaining ones with a superscript 's' are additional quantities incorporating the transverse shear deformation effects. Upon employing the Hamilton's principle, the three variationally consistent equilibrium equations of the beam are obtained as

$$N^{c}_{x,x} = (\rho_0 u + \rho_{01} u_1 - \rho_1 w_{,x})_{,tt},$$
$$M^{c}_{x,xx} = (\rho_1 u_{,x} + \rho_{11} u_{1,x} + \rho_0 w - \rho_2 w_{,xx})_{,tt} + N^{e}_{x} w_{,xx}, \tag{11a}$$
$$M^{s}_{x,x} - Q^{s}_{x} = (\rho_{01} u + \rho_{02} u_1 - \rho_{11} w_{,x})_{,tt},$$

where $_{,tt}$ denotes time derivatives and the ρ's are defined as

$$\rho_i = \int_{-h/2}^{h/2} \rho z^i dz, \qquad (i = 0,1,2),$$
$$\rho_{jm} = \int_{-h/2}^{h/2} \rho z^j f_j^m dz, \qquad (j = 0,1;\ m = 1,2) \tag{11b}$$

Moreover, the following sets of boundary conditions at the edges of the beam are obtained by application of Hamilton's principle

at x=0, a

either u or N^{c}_{x} prescribed,

either w or $M^{c}_{x,x}$ prescribed,

either $w_{,x}$ or M^{c}_{x} prescribed

either u_1 or M^{s}_{x} prescribed (12)

2.3 The continuity conditions for the symmetric cross-ply beams

By suitable changing the previous shape functions given in Eq. (7), the continuity of transverse shear strain can be satisfied. Details of this manipulation are given in the previous works (Aydogdu 2005, 2006a, 2006b). Only the final form of the new shape function that satisfies the transverse continuity conditions is described here.

$$\Phi(z) = A_\chi \varphi(z) + B_\chi \tag{13}$$

where

$$A_\chi = \frac{Q_{55}^{(\chi\mp1)} \phi'^{(\chi\mp1)}(z_\chi)}{Q_{55}^{(\chi)} \phi'^{(\chi)}(z_\chi)} A_{\chi\mp1}, \qquad A_0 = 1,$$

$$B_\chi = B_{\chi\mp1} + \varphi(z_\chi)(A_{\chi\mp1} - A_\chi), \qquad B_0 = 0.$$

By substituting Eq. (13) into Eq. (3), the following displacement field, which satisfies the transverse continuity conditions, is obtained.

$$U^{(\chi)}(x,z;t) = u^{(0)} - z w_{,x} + \left[A_\chi \phi^{(k)}(z) + B_\chi\right] u_1^{(0)},$$
$$W^{(\chi)}(x,z;t) = w. \tag{14}$$

2.4 The Ritz solution for the vibration problem of cross-ply beams with various boundary conditions

The Ritz method (Kantrovich and Krylov 1964) is a variational approach that requires the expansion of the unknown functions of the displacement components in infinite series form. By taking a sufficient number of terms in the series, it is possible to approach the exact solution of the problem considered. This method was used in past studies and related references can be found in the previous papers (Aydogdu 2005, 2006a, 2006b). A short review of this method is given below:

Upon defining the natural coordinate $\xi=x/a$, the following simple algebraic polynomials can be written

$$u(\xi,t)=\sum_{i=i_0}^{I} A_i X_i(\xi)\sin\omega t,$$
$$u_1(\xi,t)=\sum_{p=p_0}^{P} D_p X_p(\xi)\sin\omega t, \tag{15}$$
$$w(\xi,t)=\sum_{m=m_0}^{M} C_m X_m(\xi)\sin\omega t,$$

where the polynomial is defined as

$$X_f=\xi^f(\xi-1)^B, \qquad f=i,p,m \tag{16}$$

and A_i, D_p and C_m are unknown undetermined coefficients. The exponent B values are chosen according to the type of the boundary conditions imposed at the edges of the beam. The values of B=0, 1 and 2 correspond to free, simply supported and clamped edge conditions respectively (Narita 2000). Three different boundary conditions are considered in this study, namely: Hinged-Hinged (H-H), Clamped-Clamped (C-C) and Clamped-Free (CF). Here, the first and second capital letter denote the boundary conditions at ξ=0 and ξ=1, respectively. Related kinematic boundary conditions and selection of the starting indices of the series in Eq. (15) are given in the previous papers by the author (Aydogdu 2005, 2006a, 2006b). The free boundary conditions are approximately satisfied by means of the Ritz method.

Application of the Ritz method requires the kinetic energy and the strain potential energy functions of the cross-ply composite beam. The strain energy of the cross-ply composite beam can be written in terms of the middle surface displacement as follows

$$\begin{aligned} U_{s\max}=\frac{1}{2}\int_0^b\int_0^a \{&A_{11}u_{,x}^2-2B_{11}w_{,xx}u_{,x}+2E_{11}u_{1,x}u_{,x}+D_{11}w_{,xx}^2\\ &-2F_{11}u_{1,x}w_{,xx}+H_{11}u_{1,x}^2+A_{55}u_1^2\}dxdy \end{aligned} \tag{17}$$

and the kinetic energy of the cross-ply beam can be written in the following form

$$T_{\max}=\frac{1}{2}\int_V \rho\omega^2\left(u^2-2zuw_{,x}+2u\phi u_1+z^2w_{,x}^2-2z\phi u_1w_{,x}+\phi^2u_1^2+w^2\right)dV \tag{18}$$

Upon inserting the displacements and their derivatives from Eq. (15) into the strain and kinetic energy expressions given in Eqs. (17) and (18) and minimizing the functional ($U_{s\max}$-$T_{\max}$) with respect to the coefficients of the displacement functions, a set of simultaneous algebraic equations, in terms of the unknown coefficients given by Eq. (15) can be obtained. The upper limits of series in Eq. (15) are chosen to be equal, i.e., $I=P=M$ so that $3P^2$ equations are obtained. These equations can be written as a generalized eigenvalue problem

$$\{[K]-\lambda^2[M]\}\{\Delta\}=0 \tag{19}$$

where K and M are the stiffness and the inertia matrices, respectively, and Δ is the column vector of unknown coefficients of Eq. (15). The eigenvalues (λ) for which the determinant of coefficient matrix of Eq. (19) is zero, leads to the nondimensional free vibration frequency parameter. Dimensionless frequency parameter is defined as $\lambda=(\omega a^2/h)(\rho/E_2)^{1/2}$.

3. Numerical results

The elastic constants of SWCNTs are taken from the analytical results of Popov *et al*. (2000). The Young's modulus and the Poisson's ratio of the matrix (polystyrene) are E_m=1.9 GPa and ν_m=0.3 respectively. In the first part of the present study engineering constants of the SWCNT reinforced composite material are calculated using Eqs. (2)-(3) and the material constants of the matrix and results are depicted in Fig. 1. Three different CNT radii (R) are used: R=0.2 nm, 0.6 nm and 2.45 nm. It can be observed from Fig. 1 that, for small CNT diameter, the longitudinal and shear moduli of the composite material increase with increasing volume fraction, whereas the Poisson's ratio decreases.

In the formulation presented in Section 2.2, the direct use of the shape functions given in Eq. (7) for the PSDBT, FSDBT, and GESDBT violates the continuity of the interlaminar stress through the thickness of the beam. However, by suitable modifications of the shape functions, as described in Section 2.3, the continuity conditions through the thickness of the beam are satisfied for symmetric cross-ply lay-ups.

Convergence studies carried out for the fundamental frequency parameter λ of the symmetric and antisymmetric cross-ply beams were given in the previous studies (Aydogdu 2005, 2006a); details are not given here due to space limitations. Results from such studies are in good agreement with fundamental frequency values recommended by several codes (Chandrashekhara *et al*. 1990, Khdeir and Reddy 1994, Chen *et al*. 2003). All of the results presented in this work calculated using eight terms (P=8) in Eq. (15).

After verifying the accuracy and convergence of the Ritz analysis, the fundamental frequency of the symmetric cross-ply (0°/90°/0°) beams are computed using different shear deformation theories for H-H boundary conditions and for different material properties and aspect ratios (Table 1). Good agreement between different theories is observed for the fundamental frequency. The differences between the frequencies predicted with the different theories increase with decreasing *a/h* ratio.

After comparison of the theories the fundamental frequency parameters are calculated for different boundary conditions, CNT diameter, volume fraction and ascpect ratio (Fig. 2-Fig. 4). All of the remaining results will be given using Aydogdu (2009) model (GESDBT). From these figures the following conclusions can be drawn: The frequency parameter increases with increasing volume fraction. This increase is more pronounced for the high aspect ratio (*a/h*). The frequency parameter is insensitive to the volume fraction for *a/h*=5 for H-H and C-C boundary

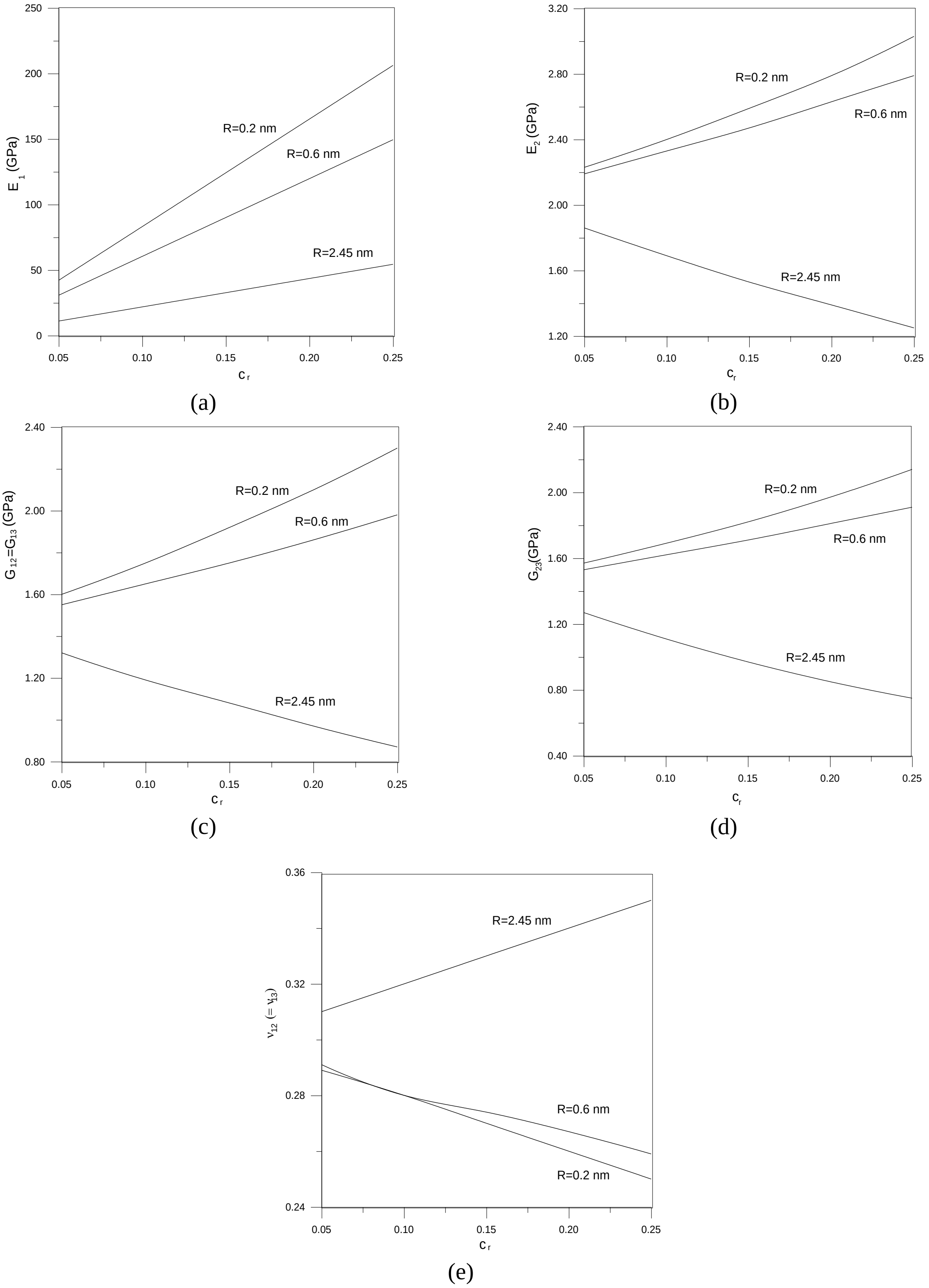

Fig. 1 Variation of the mechanical properties of the SWCNT reinforced composites with the fiber volume fraction

Table 1 Comparison of the fundamental frequency parameter for the different shear deformation theories for ($0^o/90^o/0^o$), R=0.2 nm and hinged-hinged boundary conditions

L/h	c_r=0.05			c_r=0.25		
	Reddy	Aydogdu	FOSDT	Reddy	Aydogdu	FOSDT
5	8.608	8.589	8.561	11.023	11.021	10.874
10	10.888	10.874	10.873	16.874	16.822	16.841
20	11.831	11.826	11.827	20.899	20.868	20.896
50	12.145	12.144	12.145	22.685	22.679	22.686
100	12.192	12.192	12.192	22.981	22.979	22.981

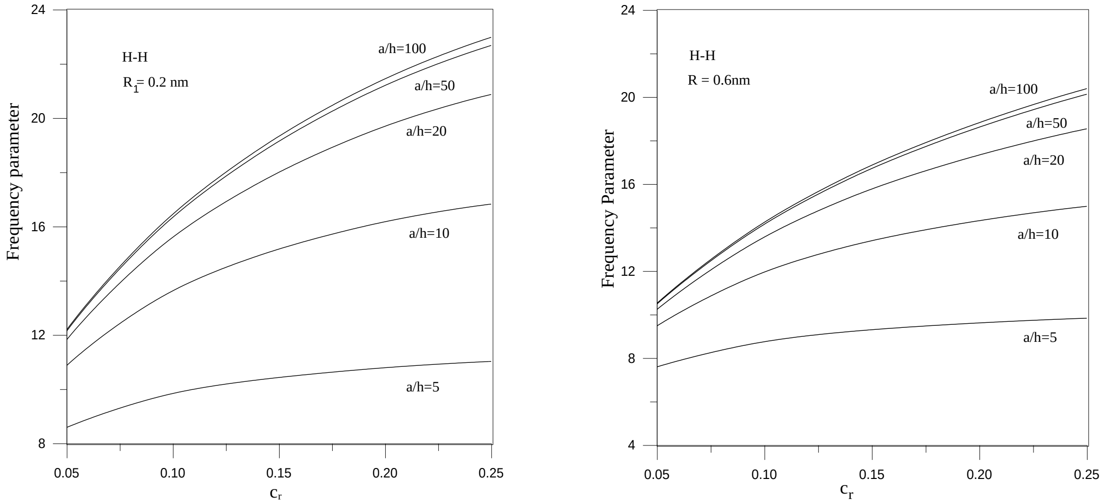

Fig. 2 Variation of the frequency parameter of the CNT reinforced composite beams with the volume fraction for H-H boundary conditions

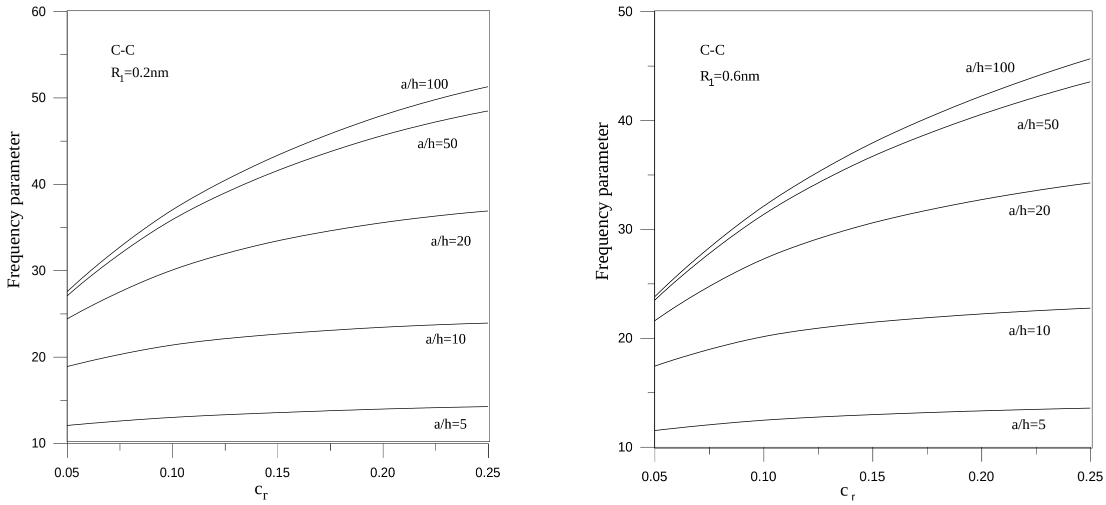

Fig. 3 Variation of the frequency parameter of the CNT reinforced composite beams with the volume fraction for C-C boundary conditions

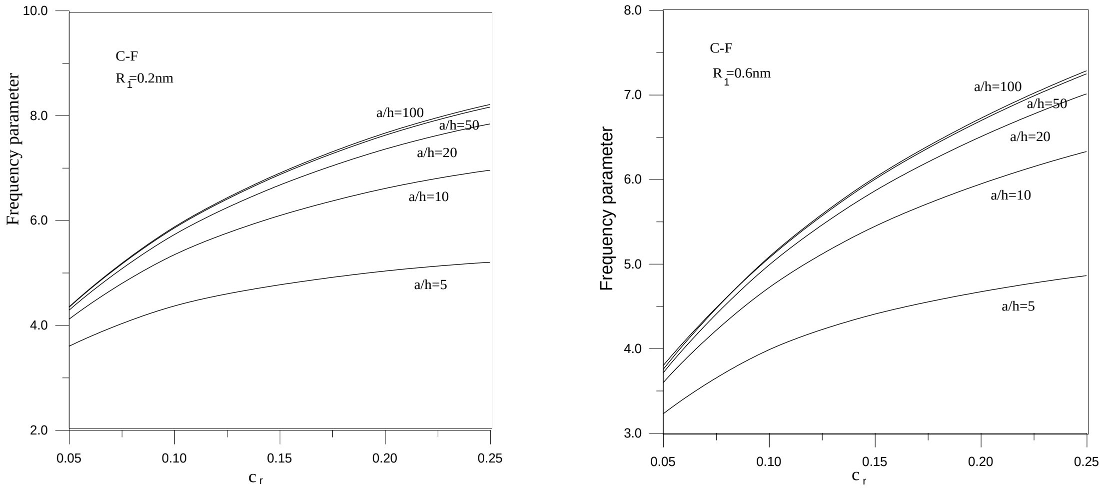

Fig. 4 Variation of the frequency parameter of the CNT reinforced composite beams with the volume fraction for C-F boundary conditions

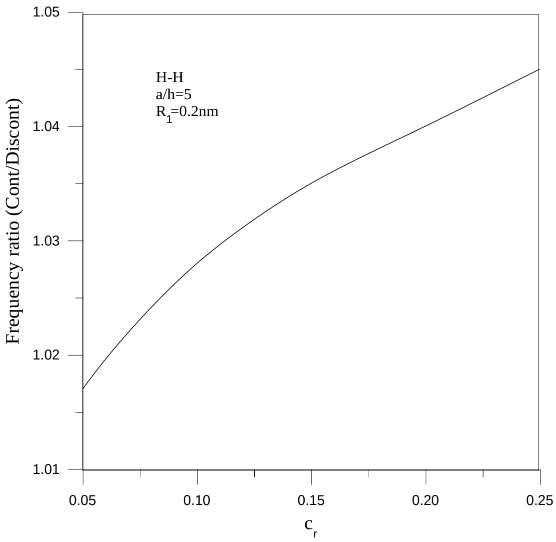

Fig. 5 Variation of the frequency ratio (Cont/Discont) with the CNT volume fraction for H-H boundary condition

conditions. The highest frequency parameter is obtained for the smallest CNT diameter (R=0.2 nm). Higher frequencies are obtained for fixed-fixed and lower frequencies for clamped-free boundary conditions. Higher frequencies are obtained for present CNT reinforced composite beams when compared with conventional composites Aydogdu (2005).

In order to see the effect of the transverse shear stress continuity, the frequency ratios are given in Fig. 5 for the continuous and the discontinuous stress case. The difference between the two cases increases with increasing volume fraction. Since the continuity effects are important for the high anisotropy and small aspect ratios, a/h=5 and R=0.2 nm are chosen in the example problem. Smaller differences are observed for the examples using other aspect ratios and CNT radii.

4. Conclusions

In the present study, vibration of the aligned Carbon nanotube (CNT) reinforced composite beams is analyzed using the shear deformable composite beam theories. The Ritz method with the algebraic polynomials is used to solve the free vibration problems of composite beams. The Mori-Tanaka method is used to find the composite beam properties. It is found that, the frequency parameter increases with increasing volume fraction. This increase is more pronounced for high aspect ratios (*a/h*). The frequency parameter is insensitive to the volume fraction for *a/h*=5 for H-H and C-C boundary conditions. The highest frequency parameter is obtained for the smallest CNT diameter (*R*=0.2 nm). Higher frequencies are obtained for fixed-fixed and lower frequencies for clamped-free boundary conditions. Effect of stacking sequence can be investigated in the future studies.

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