A new refined nonlocal beam theory accounting for effect of thickness stretching in nanoscale beams

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Abstract. In this paper, a simple and refined nonlocal hyperbolic higher-order beam theory is proposed for bending and vibration response of nanoscale beams. The present formulation incorporates the nonlocal scale parameter which can capture the small scale effect, and it considers both shear deformation and thickness stretching effects by a hyperbolic variation of all displacements across the thickness without employing shear correction factor. The highlight of this formulation is that, in addition to modeling the displacement field with only two unknowns, the thickness stretching effect ($\varepsilon \neq 0$) is also included in the present model. By utilizing the Hamilton’s principle and the nonlocal differential constitutive relations of Eringen, the equations of motion of the nanoscale beam are reformulated. Verification studies demonstrate that the developed theory is not only more accurate than the refined nonlocal beam theory, but also comparable with the higher-order shear deformation theories which contain more number of unknowns. The theoretical formulation proposed herein may serve as a reference for nonlocal theories as applied to the static and dynamic responses of complex-nanobeam-system such as complex carbon nanotube-system.

Keywords: nonlocal theory; stretching effect; nanobeam

1. Introduction

Recent experimental results have demonstrated a significant size influence in mechanical characteristics when the dimensions of the structure become small. The local continuum models lack the capability of capturing such effects since they do not incorporate any internal length scale. Thus, these models are expected to fail when the structure size becomes comparable with the internal length scale(s) of the material. This motivated many authors to propose beam/plate theories based on size-dependent continuum models which consider the small scale influences. The nonlocal elasticity theory developed by Eringen (1972, 1983) is one of the promising size-dependent continuum models. Contrary to the local continuum models which suppose that the stress at a point is a function of strain at that point, the non-classical elasticity theory considers that...
the stress at a point is a function of strains at all points in the continuum. Thus, the small scale parameter is introduced through the employment of constitutive equations.


In recent years, researchers proposed some shear deformation theories to study bending, buckling and vibration behaviors of structures (Bellifa, Benrahou et al. 2016, Tounsi, Houari et al. 2016, Bourada, Amara et al. 2016, Houari, Tounsi et al. 2016, Ait Yahia, Ait Atmane et al. 2015, Ait Amar Meziane, Abdelaziz et al. 2014, Zidi, Tounsi et al. 2014, Bouderba, Houari et al. 2013, Tounsi, Houari et al. 2013d). In addition, the stretching thickness effect was studied by several authors to show its importance on mechanical behavior of structures (Bennoun, Houari et al. 2016, Bourada, Kaci et al. 2015, Hamidi, Houari et al. 2015, Belabed, Houari et al. 2014, Hebali, Tounsi et al. 2014). Recently, many papers have been published concerning with analysis of nanostructures. Among them, Ahouel, Houari et al. (2016) examined size-dependent mechanical response of functionally graded trigonometric shear deformable nanobeams including neutral surface position concept. Ebrahimii and Barati (2016) presented an exact solution for buckling analysis of embedded piezoelectromagnetically actuated nanoscale beams. Bououara, Benrahou et al. (2016) developed a nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation. Eltaher, Khater et al. (2016) investigated the static stability of nonlocal nanobeams using higher-order beam theories.

In the present work, an analytical solution to the bending and vibration analyses of nanoscale beams is presented by proposing a novel nonlocal shear and normal deformation beam theory, which is compared with the predictions of other theories available in the literature. Just two unknown displacement functions are employed in the present model against four or more unknown displacement functions utilized in the corresponding ones. The effects due to small scale, transverse shear and thickness stretching are all included. The small scale influence is considered by utilizing the nonlocal constitutive relations of Eringen, while the shear and normal deformations effects are captured using the hyperbolic shear deformation theory (Zenkour 2013, Bourada et al. 2015). Based on the nonlocal constitutive relations of Eringen, equations of motion of nanoscale beams are obtained by employing Hamilton’s principle. Analytical solutions for deflection and natural frequency are presented for simply supported nanoscale beams, and the obtained results are compared with the existing solutions to check the accuracy of the present formulation.

2. Nonlocal beam model with thickness stretching effect

2.1 Kinematics
The displacement field of the hyperbolic shear deformation theory is proposed based on the supposition that the transverse shear stress vanishes on the top and bottom surfaces of the beam and is nonzero elsewhere. The displacement field is considered as (Bourada, Kaci et al. 2015)

\[
\begin{align*}
u(x, z, t) &= 0 \\
w(x, z, t) &= w_0(x, t) + g(z) \phi(x, t)
\end{align*}
\]

where, \( w_0 \) is the displacement of the middle surface along the axis \( z \); and the additional displacement \( \phi \) accounts for the effect of normal stress (thickness stretching effect). In this work, the shape functions \( f(z) \) and \( g(z) \) are taken based on the hyperbolic function proposed by Zenkour (2013)

\[
f(z) = h \sinh \left( \frac{z}{h} \right) - \frac{4z^3}{3h^2} \cosh \left( \frac{1}{2} \right), \quad \text{and} \quad g(z) = \frac{1}{12} f'(z)
\]

The linear strain relations associated with the displacement field in Eq. (1) are

\[
\begin{align*}
e_x &= z k_x + f(z) \eta_x \\
\gamma_{xz} &= \left[ f'(z) + g(z) \right] \eta_x^0 \\
\varepsilon_z &= g'(z) \varepsilon_z^0
\end{align*}
\]

where the prime denotes differentiation with respect to \( z \) and, \( k_x, \eta_x, \gamma_{xz}^0, \varepsilon_z^0 \) are be defined

\[
k_x = - \frac{\partial^2 w}{\partial x^2}, \quad \eta_x = \frac{\partial^2 \phi}{\partial x^2}, \quad \gamma_{xz}^0 = \frac{\partial \phi}{\partial x}, \quad \varepsilon_z^0 = \phi
\]

### 2.2 Equations of motion

Hamilton’s principle is employed to determine the equations of motion

\[
\int_{t_0}^{t_2} \left( \delta U + \delta V - \delta K \right) dt = 0
\]

where \( U, K \) and \( V \) represent the strain energy, kinetic energy and the work done by external forces, respectively.

The variation of the strain energy can be expressed as

\[
\delta U = \int_{0}^{L/2} \int_{-h/2}^{h/2} \left( \sigma_x \delta \varepsilon_x + \sigma_z \delta \varepsilon_z + \tau_{xz} \delta \gamma_{xz} \right) \, dz \, dx
\]

\[
= \int_{0}^{L} \left[ N \frac{d\delta u_0}{dx} + M \frac{d^2 \delta w_0}{dx^2} - M \frac{d^2 \delta w_0}{dx^2} + P \frac{d^2 \delta \varphi}{dx^2} + N \delta \varphi + Q \frac{d\delta \varphi}{dx} \right] \, dx
\]

where \( M, P, N, \) and \( Q \) are the stress resultants defined as
can be expressed as

\[ (M, P) = \int_{-h/2}^{h/2} (1, z, f(z))\sigma_z dz, \quad N_z = \int_{-h/2}^{h/2} \sigma_z g'(z) dz, \quad Q = \int_{-h/2}^{h/2} \tau_{xz} [f'(z) + g(z)] dz \]  

(7)

The variation of work done by externally transverse loads \( q \) can be expressed as

\[ \delta V = -\int_0^L q \delta w dx \]  

(8)

The variation of the kinetic energy is obtained as

\[ \delta K = \int_0^L \int_{-h/2}^{h/2} \rho(z) \left[ \ddot{u} \delta \phi + \dot{w} \delta \dot{w} \right] dz dx \]

\[ = \int_0^L \left[ t_0 \ddot{w}_0 \delta \dot{w}_0 + J_0 \left[ \ddot{w}_0 \delta \phi + \phi \delta \dot{w}_0 \right] \right] dx \]

\[ + I_2 \left( \frac{d\dot{w}_0}{dx} \delta \ddot{w}_0 - J_2 \left( \frac{d\dot{w}_0}{dx} \frac{d\delta \phi}{dx} + \frac{d\phi}{dx} \frac{d\delta \dot{w}_0}{dx} \right) \right) + K_0 \ddot{\phi} \delta \phi + K_2 \left( \frac{d\phi}{dx} \frac{d\delta \dot{w}_0}{dx} \right) dx \]

(9)

Where dot-superscript convention indicates the differentiation with respect to the time variable \( t \); and \( (I, J, K) \) are mass inertias defined as

\[ (I_0, I_2) = \int_{-h/2}^{h/2} \left( t, z^2 \right) \rho(z) dz \]  

(10a)

\[ (J_0, J_2) = \int_{-h/2}^{h/2} \left( g, zf \right) \rho(z) dz \]  

(10b)

\[ (K_0, K_2) = \int_{-h/2}^{h/2} \left( g^2, f^2 \right) \rho(z) dz \]  

(10c)

Substituting the expressions for \( \delta U, \delta V, \) and \( \delta K \) from Eqs. (6), (8), and (9) into Eq. (5) and integrating by parts, and collecting the coefficients of \( \delta w_0 \) and \( \delta \phi \), the following equations of motion of the beam are obtained

\[ \delta w_0 : \frac{d^2 M}{dx^2} + q = I_0 \ddot{w}_0 + J_0 \ddot{\phi} - I_2 \frac{d^2 \ddot{w}_0}{dx^2} + J_2 \frac{d^2 \delta \dot{w}_0}{dx^2} \]  

(11a)

\[ \delta \phi : \frac{d^2 P}{dx^2} - \frac{dQ}{dx} + N \dot{\phi} = -J_0 \ddot{w}_0 - J_2 \frac{d^2 \ddot{w}_0}{dx^2} - K_0 \ddot{\phi} + K_2 \frac{d^2 \delta \dot{w}_0}{dx^2} \]  

(11b)

### 2.3 Constitutive relations

The nonlocal theory considers that the stress at a point is related not only on the strain at that point but also on strains at all other points of the body. According to Eringen (1972, 1983), the
nonlocal stress $\sigma$ at a point is expressed as

$$\left(1 - \mu \nabla^2\right)\sigma = t$$

where $\nabla^2$ is the Laplacian operator, and $t$ is the classical stress. $t = e_0 a$ is the scale-effect parameter where $e_0$ is a material constant experimentally predicted, and $a$ is an internal characteristic length (e.g., lattice parameter, molecular diameter, granular distance). For one dimensional beam element with considering thickness stretching effects, the nonlocal constitutive equation, Eq. (12), can be represented by

$$1 - \mu \frac{d^2}{dx^2}\sigma_s = Q_{11}\varepsilon_s + Q_{13}\varepsilon_z$$

$$1 - \mu \frac{d^2}{dx^2}\tau_{xz} = Q_{55}\gamma_{xz}$$

$$1 - \mu \frac{d^2}{dx^2}\sigma_z = Q_{13}\varepsilon_s + Q_{33}\varepsilon_z$$

Transforming the local stress resultants defined in Eq. (7), to nonlocal domain using the differential operator of Eringen, Eqs. (13), we obtain

$$1 - \mu \frac{d^2}{dx^2}M = \int \left( \varepsilon_z, \varepsilon_s, f^2, \gamma^2 \right) \rho_1 dA, \quad A_s = \int \varepsilon_z^2 Q_{55} dA, \quad \left( D_{st}, H_{st} \right) = \int g^1 \left( z, f \right) Q_{13} dA,$$

By substituting Eq. (14) into Eq. (11), the nonlocal equations of motion can be expressed in terms of displacements ($w_0, \phi$) as

$$-D \frac{\partial^4 w_0}{\partial x^4} + D_s \frac{\partial^4 \phi}{\partial x^4} + D_{st} \frac{\partial^2 \phi}{\partial x^2} + q - \mu \frac{\partial^2 q}{\partial x^2} = I_0 \left( \frac{d^2 w_0}{dx^2} - \mu \frac{d^4 w_0}{dx^4} \right) - I_2 \left( \frac{d^2 \phi}{dx^2} - \mu \frac{d^4 \phi}{dx^4} \right) + J_0 \left( \frac{d^2 \phi}{dx^2} - \mu \frac{d^4 \phi}{dx^4} \right) + J_2 \left( \frac{d^2 \phi}{dx^2} - \mu \frac{d^4 \phi}{dx^4} \right)$$

This represents the nonlocal equations of motion for a beam element with thickness stretching effects.
\[-D_1 \frac{\partial^4 w_0}{\partial x^4} - D_2 \frac{\partial^2 w_0}{\partial x^2} + H_1 \frac{\partial^3 \varphi}{\partial x^3} + 2H_2 \frac{\partial^2 \varphi}{\partial x^2} - A \frac{\partial^2 \varphi}{\partial x^2} + F_{st} \varphi = -J_0 \left( \frac{d^2 \tilde{w}_0}{dx^2} - \mu \frac{d^4 \tilde{w}_0}{dx^4} \right) \]  

(16b)

3. Analytical solution of simply supported nanobeam

In this study, analytical solutions are given for simply supported isotropic nanobeams for bending and free vibration.

The following displacement field satisfies boundary conditions and governing equations.

\[
\begin{bmatrix} w_0 \\ \varphi \end{bmatrix} = \sum_{n=1}^{\infty} \begin{bmatrix} W_n e^{i \omega x} \sin \beta x \\ \phi_n e^{i \omega x} \sin \beta x \end{bmatrix}
\]  

(17)

where \( W_n \) and \( \phi_n \) are arbitrary parameters to be determined, \( \omega \) is the eigenfrequency associated with \( n \)th eigenmode, and \( \beta = \frac{n \pi}{L} \). The transverse load \( q \) is also expanded in the Fourier sine series as

\[ q(x) = \sum_{n=1}^{\infty} Q_n \sin (\beta x), \quad Q_n = \frac{2}{L} \int_0^L q(x) \sin (\beta x) dx \]  

(18)

The Fourier coefficients \( Q_n \) associated with some typical loads are given

\[ Q_n = q_0, \quad n = 1 \text{ for sinusoidal load}, \]  

(19a)

\[ Q_n = \frac{4q_0}{n \pi}, \quad n = 1,3,5 \ldots \text{for uniform load}, \]  

(19b)

\[ Q_n = \frac{2q_0 \sin \frac{n \pi}{2}}{L}, \quad n = 1,2,3 \ldots \text{for point load } Q_0 \text{ at the midspan}, \]  

(19c)

Substituting the expansions of \( w_0, \varphi, \) \( \varphi' \) and \( q \) from Eqs. (17) and (18) into Eq. (16), the closed form solutions can be obtained from the following equations

\[
\begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} - \lambda \omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{bmatrix} W_n \\ \phi_n \end{bmatrix} = \lambda \begin{bmatrix} Q_n \\ 0 \end{bmatrix},
\]  

(20)

where

\[ S_{11} = D \beta^4, S_{12} = -D \beta^4 + D_2 \beta^2, S_{22} = H_2 \beta^2 - 2H_2 \beta^2 + A \beta^2 + F_{st} \beta, \lambda = 1 + \mu \beta^2 \]

\[ m_{11} = I_0 + I_2 \beta^2, m_{12} = I_0 - J_2 \beta^2, m_{22} = K_0 + K_2 \beta^2. \]  

(21)
4. Numerical results

Through this section, the effect of thickness stretching in nanobeam, nonlocality effect and slenderness ratios on the deflections and natural frequencies of the nanobeam will be discussed. The obtained results are compared with those predicted using the Euler-Bernoulli beam theory (EBT), Timoshenko beam theory (TBT), Reddy’s beam theory (RBT) and the model of Berrabah, Tounsi et al. (2013) for a wide range of nonlocal parameter and slenderness ratio. For all computations, the shear correction factor and Poisson’s ratio are considered as 5/6 and 0.3, respectively. The length of nanobeam \( L \) is supposed to be 10 nm. A conservative estimate of the nonlocal scale parameter \( 0 \leq \epsilon_0 \alpha \leq 2 \) nm for single-walled carbon nanotubes (SWCNTs) is proposed by Wang (2005). Hence, in this work, the nonlocal parameter is taken as \( \mu=(\epsilon_0 \alpha)^2=0,1,2,3 \) and 4 nm to examine the nonlocal effects on the responses of nanobeam. For convenience, the following non-dimensional quantities are employed:

\[-w = 100w \frac{EI}{q_0L^4} \text{ for uniform load;}
\]

\[-\omega = \omega L^2 \left( \frac{I_0}{EI} \right) \text{ frequency parameter;}
\]

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<th>( L/h )</th>
<th>( \mu ) (nm(^2))</th>
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<th>TBT</th>
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<th>Tounsi, Benguediab et al. (2013a) (( \epsilon_0 \neq 0 ))</th>
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Table 1 illustrates the variation of the non-dimensional maximum deflections $\bar{w}$ with respect to nonlocal scale parameter, proposed theories, and slenderness ratios. A simply supported nanobeam subjected to uniform load is considered in this example and the calculated values are obtained using 100 terms in series in Eqs. (17) and (18). The predicted results are compared to those given by the Euler-Bernoulli beam theory (EBT), Timoshenko beam theory (TBT), Reddy’s beam theory (RBT), Sinusoidal beam theory (SBT) of Berrabah, Tounsi et al. (2013) and the theory developed by Tounsi, Benguediab et al. (2013a). For all theories, it is noted that the deflection increases as the nonlocal scale parameter increases at a specified slenderness ratio. Moreover, for high slenderness ($L/h=100$) ratio, all theories are approximately identical in predicting the deflection, which confirms the accuracy of the simple Euler-Bernoulli model in the case of thin nanoscale beams. However, the discrepancy between EBT and other theories is noticeable for a moderately thick beam ($L/h=10$). On the other hand, the results predicted by employing the TBT coincide with those obtained using higher-order theories suggesting the accuracy of utilizing TBT for the case of moderately thick beams. It can be seen that the results from TBT, RBT and SBT due to ignoring the thickness stretching effect ($\varepsilon_z=0$) are slightly overestimate when comparing with those from the present theory (quasi-3D, $\varepsilon_z\neq 0$). This effect is more pronounced on thick beams ($L/h=5$). Noted that the present model has only three unknowns as in the case of TBT, RBT and SBT, while the number of unknowns in quasi-3D (Tounsi, Benguediab et al. 2013a) is four. Also, the present theory does not required shear correction coefficients as in the case of TBT.

Table 2 Dimensionless fundamental frequency $\bar{\omega}$ of simply supported nanobeam

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The non-dimensional frequency $\bar{\omega}$ of a simply supported nanoscale beam are shown in Tables 2 and 3 for various values of scale parameter $\mu$ and four different values of slenderness ratio ($L/h=5, 10, 20, 100$) based on analytical Navier solution technique. It can be concluded from these
results that an increase in nonlocal parameter gives rise to a decrement in the frequency. In addition, it is seen that the $\bar{\sigma}$ increases by increasing slenderness ratio ($L/h$) and it can be stated that nonlocality factor has a notable influence on the frequency and especially at the higher vibration modes (see Table 3). Our results are in good agreement with those obtained by Berrabah, Tounsi et al. (2013) for EBT, TBT, RBT, and SBT. However, it can be seen that, the inclusion of thickness stretching effect (i.e., $e_{z}\neq0$) leads to a slight increase of frequency.

Fig. 1 demonstrates the variation of deflection and frequency ratios of nanobeam with the slenderness ratio ($L/h$). In this example, the deflection, and frequency ratios are defined as the ratios of those computed by present formulation to the correspondences computed by EBT where the shear deformation effect is neglected. Observing this figure, it is easily deduced that, the influence of slenderness ratio is to decrease the natural frequencies and increase the deflections, and this effect is considerable for thick beams at higher vibration modes (see Fig. 2). This demonstrates that the slenderness ratio effect results in a reduction of the beam stiffness. Also it can be concluded from the results of the Fig. 1 that the present nonlocal model is capable to produce very accurate results compared with the nonlocal theory developed by Tounsi, Benguediab et al. (2013a) with higher number of unknowns.

The influence of the nonlocal scale parameter on the bending and vibration behaviors of nanoscale beam is shown in Fig. 3. The transverse displacement, and frequency ratios are defined as the ratios of those calculated by the nonlocal theory to the correspondences calculated by the local theory (i.e., $\mu=0$). This figure demonstrates a nonlinear variation of the bending and vibration responses with the nonlocal scale parameter. It can be observed that the transverse displacement ratio is greater than unity, whereas the frequency ratios are smaller than unity. It means that the local theory under-estimates the transverse displacements and over-estimates the frequencies of the nanoscale beams compared to the nonlocal one. This is due to the fact that the local model is

![Graph](image-url)
A new refined nonlocal beam theory accounting for effect of thickness stretching...  

Fig. 3 Effect of the scale parameter on the deflection, and fundamental frequency ratios for a simply supported nanobeam with $L/h=10$

Fig. 4 Effect of the aspect ratio on higher frequency ratios for a simply supported nanobeam with $e_0=1$ nm

unable to consider the small scale influence of the nanoscale beams. The difference between the local and nonlocal models is especially important for the higher modes (see Fig. 4).
5. Conclusions

A novel nonlocal thickness-stretching hyperbolic shear deformation beam theory is proposed for the bending, and dynamic behavior of nanobeams. The present theory is able to consider the small scale, shear deformation and thickness stretching influences of nanoscale beams, and respects the zero traction boundary conditions on the upper and lower surfaces of the nanoscale beam without employing shear correction coefficient. From Hamilton’s principle as well as nonlocal elasticity theory of Eringen, the nonlocal equations of motion are obtained according to the refined two-variable shear deformation beam theory and then solved via an exact analytical solution. Results demonstrate that the incorporation of thickness stretching influence makes a nanoscale beam stiffer, and hence, leads to a diminishing of transverse displacement and an increase of frequency. However, it is remarked that the consideration of the nonlocal parameter and shear deformation influences lead to an increase in the transverse displacements and a reduction of the natural frequencies of nanoscale beams.

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References


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