Analytical solution for analyzing initial curvature effect on vibrational behavior of PM beams integrated with FGP layers based on trigonometric theories

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Abstract. In the current study, the free vibrational behavior of a Porous Micro (PM) beam which is integrated with Functionally Graded Piezoelectric (FGP) layers with initial curvature is considered based on the two trigonometric shear deformation theories namely SSDBT and Tan-SDBT. The structure’s mechanical properties are varied through its thicknesses following the given functions. The curved microbeam is exposed to electro-mechanical preload and also is rested on a Pasternak type of elastic foundation. Hamilton’s principle is used to extract the motion equations and the MCST is used to capture the size effect. Navier’s solution method is selected as an analytical method to solve the motion equations for a simply supported ends case and by validating the results for a simpler state with previously published works, effects of different important parameters on the behavior of the structure are considered. It is found that although increasing the porosity reduces the natural frequency, but enhancing the volume fraction of CNTs increasing it. Also, by increasing the central angle of the curved beam the vibrations of the structure increases. Designing and manufacturing more efficient smart structures such as sensors and actuators are of the aims of this study.

Keywords: curved beam; porous materials; carbon nanotubes reinforced composites; sandwich structures; trigonometric shear deformation theory; modified couple stress theory

1. Introduction

The porous medium mechanic is a branch of physics that examines the behavior of porous materials which their pores filled with a fluid such as water. The skeletal part of porous material is called matrix, which usually is solid. Porosity is the main property of the porous materials that indicates the ratio of pores volume to that of the whole. Porosity can be measured by various methods, such as observation, using an approximate microscope, gas diffusion and also using the mercury pump.

Porous materials are used in many fields of applied sciences and engineering, such as petroleum engineering, construction engineering, rock mechanics, soil mechanics and materials engineering (Liu and Chen 2014). Also, these materials are used as sound insulation due to their high strength and low weight inland transport, railways, sea and air and because of their high energy absorption capacity (Arshid and Khoshivand 2018). Considering porosity effect in structures such as beams, plates and shells helps to gain better knowledge about their mechanical behavior.

Mechanics of porous mediums is derived from soil mechanics, in which Terzaghi, carried out specialized research in this field (Von Terzaghi 1923). He investigated a theory in which the effect of the fluid on the quasi-static deformation of the soil was accounted for. In 1935, Biot considered the mechanics of a porous material filled with fluid. His theoretical model was a solid skeleton contains pores and fluid in the pores. The concept of this model was to express that solid and fluid phases are related to each other. After Biot, the constitutive relations for porous materials were presented by Detournay and Cheng (1993). Nowadays, numerous researchers are interested in studying the mechanical behavior of porous structures. In 2004, Magnucki and Stasiwicz (2004) studied the elastic buckling of a porous beam under different loading conditions. Wattanasakulpong and Ungbhakorn (2012) studied the buckling behavior of a porous Functionally Graded (FG) beam using the Differential Quadrature Method (DQM). Bending and buckling behaviors of the FG beams presented by Chen et al. (2015). They used the Timoshenko beam model to account shear deformation effect and achieved the results via the Ritz method and consider the effect of different parameters on them. Ait Attmane et al. (2017) considered the effects of the thickness stretching and porosity on the mechanical behavior of the FG beam which was located on an elastic foundation and used Navier’s method to solve the governing equations. Ebahimi and Mokhtari (2015) presented the vibration behavior of a rotating FG porous beam and solved the equations numerically via DQM. Ebahimi and Jafari (2016) analyzed the thermomechanical vibrations of the FG porous beam under different temperature loading using a semi-analytical differential conversion technique. Ebahimi and Barati (2017) presented a high-order modified beam model for considering the vibration of the viscoelastic
nanocrystalline silicon nanobeam with porosity. They observed that damping frequencies of a nanocrystalline nanobeam were significantly influenced by the grain size, grain rotations, porosities, interface, damping coefficient, surface energy, nonlocality and also structural damping. Kitipornchai et al. (2017) researches about mechanical behavior of FG porous beam which was integrated by graphene platelets and proved that graphene platelets are very effective as the reinforcement. Arshid et al. (2019a) considered vibration of saturated porous plates based on classical, first-order and third-order shear deformation theories. In another work, Arshid and Khorshidvand (2018) presented the results about the effect of piezoelectric actuators on the natural frequencies of a porous plate which was integrated by piezoelectric layers. They showed the porosity increasing, reducing the natural frequency.

Composites are multi-component materials whose properties are generally better than that of each component. Each composite consists of a matrix phase and one or more reinforcement phases. Should be noted that the components of a composite do not chemically combine in such a way that these components fully maintain their chemical and natural properties and thus creating a definite common interface between the components (McEvoy and Correll 2015). Composites have several advantages, therefore they are used widely nowadays. Composite materials may be used in fuel tanks and tubes, military industries, automobile industries, marine structures, construction industry, sports equipment, medical equipment, and so on (Tayeb et al. 2020).

In recent years, by the rapid advancement in sciences and technologies, nanotubes are used for industrial applications (Han and Elliott 2007). In the past two decades, their usage has accelerated with the discovery of Carbon Nanotubes (CNTs) by Iijima (1991). CNTs have unique mechanical, magneto-electrical, and also chemical properties. The high rigidity and stiffness to weight ratio, electrical conductivity and high-temperature strength are among these characteristics (Ajayan and Zhou 2001). CNTs are used as reinforcement of composites and would improve their properties significantly. CNTs are divided into single-walled and multi-walled types. Single-Walled Carbon Nanotubes (SWCNTs) consist of carbon and a simple structure only. Some prediction suggests that SWCNTs can be conductive or semiconductor (Anumandla and Gibson 2006). The high electrical conductivity depends on the exact geometry of carbon atoms. SWCNTs also are divided into three major categories according to the arrangement of the carbon atoms in the tube section. Chiral and armchair, whose have a metallic property and zigzag that are semiconductive. In zigzag and armchair structures, the honeycomb rows in the nanotube wall are perpendicular to the nanotube axis. This structure is also called the spiral structure (Lunhui et al. 2008).

Mechanical analysis of Carbon Nanotubes Reinforced Composites (CNTRCs) is presented by various researchers. Thostenson and Chou (2003) modeled the elastic properties of CNTRCs in 2003. They investigated the effect of the size and structure of CNTs based on the elastic characteristics of composite base nanotubes, which was their main goal. Zhu et al. (2012) analyzed the static deflection and free vibration of composite plates reinforced with CNTs using the Finite Element Method (FEM) based on First-order Shear Deformation Theory (FSDT). They studied the effects of the CNTs’ failure and the edge-to-thickness ratio on bending response, natural frequencies and the shape modes of the plate. The bending behavior of the composite plate reinforced by CNTs, which was located between two piezoelectric layers under a uniform mechanical load investigated by Aliebigloo (2013). Duc et al. (2017) considered the thermal and mechanical stability of FG CNTRC truncated conical shells. They used classical shells theory and Galerkin method to obtain the results and considered the effect of the most important variants such as semi-vertex angle and elastic medium on the linear thermal and mechanical buckling load. Lei et al. (2013) investigated the buckling of FG CNTRC plates. They obtained the effective properties of nanocomposite using the Eshelby-Mori-Tanaka approach and considered different parameters and their effects on the results. Critical buckling load of a quadrilateral laminated plate which was reinforced by CNTs obtained by Malekzadeh and Shojaee (2013). Their research was about the moderately thick plate and investigated the effect of material and geometrical shape parameters.

Since the behaviors of structures in macro and small dimensions are different, and also due to the application of small scale structures such as nano and microbeams, plates and shells in measuring equipment, medicine and industries, the researchers were encouraged to analyze them in small scales. One of the first studies about the mechanical behavior of the nanostructure was provided by Eringen. He presented the nonlocal theory and studied the structures in the nanoscale (Eringen 1983, 2002). After Eringen’s studies, the study of small scales effects extended rapidly (Mehar and Panda 2019, Jafari and Ezzati 2017, Mirjavadi et al. 2020, Jafari et al. 2016, Bennamsour et al. 2019, Akbas 2018). Amir et al. (2019) used Modified Couple Stress Theory (MCST) to capture size effect for analyzing a sandwich circular plate and a microbeam, respectively which their cores were made of porous materials.

There are different methods to solve the differential equations system for analyzing the mechanical responses of engineering structures. In this article, Navier’s technique as an exact numerical method is employed which via it, the results can be extracted for simply supported boundary conditions. The main advantage of Navier’s technique is its simplicity and accuracy both together. But, there are other advanced numerical methods for instance if the analytical solution could not be adopted in some problems. FEM, DQM, meshfree, finite difference, isogeometric, etc. are from the well-known ones of numerical solution methods. For example, transient responses and natural frequencies of sandwich beams with inhomogeneous FG core were investigated by Bui et al. (2013). They used a meshfree method to obtain the results. In another study, Bui et al. (2011) presented a detailed analysis of natural frequencies of laminated composite plates using the meshfree moving Kriging interpolation method. They used Kirchhoff’s hypothesis for their investigation. DQM was used by Arshid et al. (2020a, b) to investigate two types of the microplate.
The first one was three-layered, and the second one was a porous reinforced nanocomposite. The isogeometric method was used by researchers in different studies. For instance, Yu et al. (2019a) presented isogeometric analysis of size-dependent effects for FG microbeams by a non-classical quasi-3D theory. In another study, Liu et al. (2019) considered size and surface effects on the mechanical behavior of thin nanoplates incorporating microstructures using isogeometric analysis. Moreover, Fang et al. (2019) analyzed the mechanical behavior of porous beams by an effective computational approach based on isogeometric analysis. A novel study about size-dependent quasi-3D isogeometric beam model for two-directional FG microbeams provided by Yu et al. (2019b). Based on the MCST, the non-classical theory of the Reissner-Mindlin plate was extended to capture microstructure, and thus, the size effect of cracked FG microplates via an extended isogeometric analysis-based effective approach by Liu et al. (2018). Furthermore, a computational approach based on isogeometric analysis and a non-classical simple FSDT for size effects of FG microplates was presented by Liu et al. (2017).

By reviewing the literature, the authors found out that the lack of an investigation on a three-layered curved microbeam is sensible. Therefore, the main novelty of this work is its geometric type. Selection and placement of materials are other aspects of the novelties that there is no similar study in the literature like this. Taking the small-scale effects into account via the MCST integers the novelty of the current study. The main aim of the current study is to show the effect of porous and nanocomposite materials properties such as porosity, pores’ distribution, central angle, CNTs’ volume fraction, CNTs’ distribution types, size effect and electro-mechanical preloads on vibrational behavior of the sandwich microbeam with initial curvature. Two different trigonometric shear deformation theory namely Sinusoidal Shear Deformation Beams Theory (SSDBT) and Tangential Shear Deformation Beams Theory (Tan-SDBT) are employed to describe the displacement components of the structure. Porous materials are selected as the core of the structure and its face sheets are from nanocomposites and the structure is resting on the Pasternak foundation. The mechanical properties of the three layers are varied through the thickness according to specific patterns. To the best authors’ knowledge and by reviewing the literature, there is no similar research about such a structure.

2. Basic relations

2.1 Geometry

As shown in Fig. 1, the under consideration structure is a three-layered curved beam with a curvature radius of \( R \) and the central angle \( \theta \). The length of the beam is shown by \( L \), its total thickness is shown by \( H = h_c + h_t + h_b \) in which \( h_c \), \( h_t \) and \( h_b \) are the core, top and bottom layers’ thicknesses. Noted that here, \( L = R\theta \). The core of the structure is made of FG porous materials and the face sheets are from FG-CNTRCs. The mechanical properties of the three layers are varied through their thickness according to the functions which are introduced in the following. Also, the face sheets are exposed to the electric field and the beam is laid on an elastic foundation that is modeled by the Pasternak type. The Cartesian coordinate system is employed to describe the displacements of the structure which its origin is put on the left corner of the mid-surface of the beam.

2.2 Kinematic relations

The first step to analyze the above-mentioned structure is choosing the displacement field. Since accounting the shear deformation effects leads to more accurate and reliable results, so in the current study, two different higher-order trigonometric shear deformation theories namely SSDBT and Tan-SDBT are employed to describe the displacement components of the microbeam. Based on these theories, the displacement field is expressed as Ebrahimi et al. (2017)

\[
\begin{align*}
U(x, z, t) = & \, u(x, t) - z \frac{\partial}{\partial x} w(x, t) + f(z) \phi(x, t) \\
V(x, z, t) = & \, 0 \\
W(x, z, t) = & \, w(x, t)
\end{align*}
\]

(1)

where \( U, V \) and \( W \) are the displacements of each point of the beam along the \( x, y \) and \( z \) directions, and \( u, v \) and \( w \) are those of mid-plane. Also, \( \phi \) is the rotation of the middle surface around the \( y \)-axis. In the above equations, \( f(z) \) is the shear deformation function which is defined for SSDBT and Tan-SDBT as follows (Ebrahimi et al. 2017)

\[
\begin{align*}
f(z) &= \frac{h}{\pi} \sin \left( \frac{\pi z}{2H} \right) \quad \text{For SSDBT} \\
f(z) &= \tan \left( \frac{\pi z}{2H} \right) \sec \left( \frac{\pi z}{2H} \right), \quad r = 0.03 \quad \text{For Tan – SDBT}
\end{align*}
\]

(2) and \( (3) \)

According to Von-Karman’s assumptions, the non-zero strain’s components may be written as follows (Liu and
\[
\varepsilon_{xx} = \frac{\partial u(x,t)}{\partial x} - z \frac{\partial^2 w(x,t)}{\partial x^2} + f(z) \frac{\partial \varphi(x,t)}{\partial x} + \frac{w(x,t)}{R} \]
\[
\gamma_{xz} = \left( \frac{d}{dz} f(z) \right) \varphi(x,t) - \frac{u(x,t)}{R} + \frac{z}{R} \frac{\partial w(x,t)}{\partial x} \tag{4}
\]

2.3 Constitutive law for the face sheets

As stated before, the face sheets of the structure are made of piezoelectric FG CNTRCs. Therefore, their constitutive relations are expressed as follows (Mohammadimehr et al. 2019)
\[
\begin{bmatrix}
\{\sigma_{xx}\} \\
\{\sigma_{zz}\}
\end{bmatrix} = 
\begin{bmatrix}
[Q_{11}(z)] & [0] & [E_{11}] \\
[0] & [Q_{55}(z)] & [E_{15}]
\end{bmatrix} \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{zz}
\end{bmatrix} - 
\begin{bmatrix}
[0] & [E_{11}] & [E_{15}]
\end{bmatrix} \begin{bmatrix}
\{E_x\} \\
\{E_y\}
\end{bmatrix} \tag{5}
\]
\[
\begin{bmatrix}
\{D_x\} \\
\{D_z\}
\end{bmatrix} = 
\begin{bmatrix}
[e_{31}] & [0] & [\varepsilon_{xx}] \\
[0] & [e_{15}] & [\varepsilon_{zz}]
\end{bmatrix} \begin{bmatrix}
\{E_x\} \\
\{E_y\}
\end{bmatrix} + 
\begin{bmatrix}
[0] & [S_{11}] & [0]
\end{bmatrix} \begin{bmatrix}
\{E_x\} \\
\{E_y\}
\end{bmatrix} \tag{6}
\]

in which \(\sigma_{xx}, \sigma_{zz}, E_i\) and \(D_i (i = x, z)\) are the stress and strain tensors and electric field and electric displacement vectors, respectively. Also, \(e_{ij}\) and \(S_{ij}\) are piezoelectric and dielectric permeability coefficients, respectively. Furthermore, \(Q_{ij}\) are the stiffness components that are defined as follows.

\[
Q_{11}(z) = \frac{E_{11}}{1 - \nu^2} \tag{7}
\]
\[
Q_{55}(z) = G_{12} \tag{8}
\]

In the above relations, \(E_{11}\) and \(G_{12}\) are the longitudinal Young’s elasticity and the shear moduli and the Poisson’s ratio is shown by \(\nu\).

There are methods to determine the effective properties of nanocomposites such as Mori-Tanaka, Halpin-Tsai and the ERM. Due to simplicity and accuracy, ERM is used to this aim in the current study. According to ERM, the longitudinal and shear moduli can be obtained using the following relations (Shen 2012)
\[
E_{11} = \eta_1 E_{11CNT} V_{CNT} + V_m E_m \tag{8}
\]
\[
\frac{\eta_3}{G_{12}} = \frac{V_{CNT}}{G_{12CNT} + V_m G_m} \tag{9}
\]

where \(\eta_i (i = 1, 3)\) the efficiency parameters of CNTs, which is used to increase the accuracy of ERM to make the results close to experimental ones. These coefficients are obtained by molecular dynamics. Also, \(V_{CNT}\) and \(V_m\) are the CNTs and matrix volume fractions, respectively, that their summation is equal to 1. Noted that the CNT and \(m\) superscripts denote those of CNTs and matrix, respectively. Other properties of nanocomposites such as density and electrical properties also follow the ERM as follows (Mohammadimehr et al. 2019)
\[
P_{ij} = V_{CNT} P_{ij}^{CNT} + V_m P_{ij}^m \tag{10}
\]

where \(P_i\) are one of the above-mentioned properties.

Furthermore, the Poisson’s ratio is determined using the following relation
\[
\nu_{12} = V_{CNT}^{*} V_{12}^{CNT} + V_m \nu_m \tag{11}
\]

where \(V_{CNT}^{*}\) is the volume fraction of CNTs which is defined as (Shen 2011).
\[
V_{CNT}^{*} = \frac{w_{CNT}}{w_{CNT} + \left( \frac{\rho_{CNT}}{\rho_m} - \frac{\rho_{CNT}}{\rho_m} \right) V_{CNT}} \tag{12}
\]

That \(w_{CNT}\) is the CNTs’ mass fraction, \(\rho_{CNT}\) and \(\rho_m\) are CNTs and matrix density, respectively. In this study, the distribution of CNTs through the thickness direction of the face sheets is considered in five different patterns. Therefore, for the top face sheet the CNTs are distributed according to the following functions (Mohammadimehr et al. 2019).
\[
V_{CNT}^{f} = V_{CNT}^{*} V_{UD}^{CNT} \tag{13}
\]
\[
V_{CNT}^{f} = \left[ 1 - \frac{2}{h_f} \left( z - \frac{h_c + h_l}{2} \right) \right] V_{CNT}^{*} V_{FGA} \tag{14}
\]
\[
V_{CNT}^{f} = \left[ 1 + \frac{2}{h_f} \left( z - \frac{h_c + h_l}{2} \right) \right] V_{CNT}^{*} V_{FGV} \tag{15}
\]
\[
V_{CNT}^{f} = \frac{4}{h_f} \left[ \frac{z - \frac{h_c + h_l}{2}}{2} \right] V_{CNT}^{*} V_{FGX} \tag{16}
\]

Also, for the bottom face sheet the CNTs distribution is considered as follow (Mohammadimehr et al. 2019).
\[
V_{CNT}^{b} = V_{CNT}^{*} V_{UD}^{CNT} \tag{17}
\]
\[
V_{CNT}^{b} = \left[ 1 - \frac{2}{h_b} \left( z + \frac{h_c + h_l}{2} \right) \right] V_{CNT}^{*} V_{FGA} \tag{18}
\]
\[
V_{CNT}^{b} = \left[ 1 + \frac{2}{h_b} \left( z + \frac{h_c + h_l}{2} \right) \right] V_{CNT}^{*} V_{FGV} \tag{19}
\]
\[
V_{CNT}^{b} = \frac{4}{h_b} \left[ z + \frac{h_c + h_l}{2} \right] V_{CNT}^{*} V_{FGX} \tag{20}
\]
These patterns of CNTs distribution are shown in Fig. 2, schematically.

As stated before, the face sheets are exposed to the electric field. Noted that the electric field should be defined in such a way to satisfy Maxwell’s relation. So, Arshid et al. (2019b) is

\[ E = - \nabla \phi \]  

(15)
in which \( \phi \) is the electric potential function which consists of linear and cosine terms as following

\[ \phi(x, z, t) = \frac{2z}{h_f} \phi_0 - \cos \left( \frac{\pi z}{h_f} \right) \phi(x, t) \]  

(16)

where \( \phi_0 \) is the external applied electric potential and \( h_f \) is the thickness of the face sheets.

Substituting Eq. (16) in Eq. (15), the components of the electric field are achieved as below.

\[ E_x = - \frac{\partial \phi}{\partial x} = \frac{\partial \phi_0}{\partial x} \cos \left( \frac{\pi z}{h_f} \right) \]  

\[ E_z = - \frac{\partial \phi}{\partial z} = - \frac{2}{h_f} \phi_0 - \frac{\pi}{h_f} \phi \sin \left( \frac{\pi z}{h_f} \right) \]  

(17)

2.4 Constitutive law for the core

The stress-strain relations for the FG porous core are demonstrated as follows

\[ \begin{bmatrix} \sigma_{xx} \\ \sigma_{zz} \end{bmatrix} = \begin{bmatrix} A(z) & 0 \\ 0 & G(z) \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{zz} \end{bmatrix} \]  

(18)

where \( A(z) \) and \( G(z) \) are the stiffness components which are defined as

\[ A(z) = \frac{E(z)}{1 - \nu^2}, \quad G(z) = \frac{E(z)}{2(1 + \nu)} \]  

(19)

In the above equations, \( \nu \) is the Poisson’s ratio and \( E(z) \) is the elasticity Young’s modulus which depends on thickness. In other words, the mechanical properties of the FG porous core are distributed across its thickness direction according to three different functions which are known as asymmetric, symmetric, and uniform porosity distributions. For each of the mentioned porosity distribution types, the following relations are ruled.

2.4.1 Non-symmetric porosity distribution

In this case, the pores are distributed non-symmetrically respect to the mid-surface of the core and the elasticity modulus and density variations are considered as following (Barati and Zenkour 2017)

\[ \begin{align*}
E(z) &= E_1 \left( 1 - e_1 \cos \left( \frac{\pi z}{2h_c} + \frac{\pi}{4} \right) \right) \\
\rho(z) &= \rho_1 \left( 1 - e_m \cos \left( \frac{\pi z}{2h_c} + \frac{\pi}{4} \right) \right)
\end{align*} \]  

(20)

where \( e_1 \) is the porosity coefficient which indicated the ratio of pores to bulk volume and \( e_m \) is called the mass density coefficient of the structure which they are defined as (Arshid and Khorshidvand 2018).

\[ e_1 = 1 - \frac{G_1}{G_0} = 1 - \frac{E_1}{E_0} \]  

\[ e_m = 1 - \frac{\rho_1}{\rho_0} = 1 - \sqrt{1 - e_1} \]  

(21)

Noted that the value of \( e_1/e_1 \) is between zero and 1. Also, \( E_1 \) and \( E_0 \) are Young’s modulus at \( z = -h_c/2 \) and \( z = h_c/2 \), \( \rho_1 \) and \( \rho_0 \) are the densities at \( z = -h_c/2 \) and \( z = h_c/2 \), respectively.

2.4.2 Symmetric porosity distribution

Unlike the previous porosity distribution, for this type of as can be expected, the pores are distributed symmetrically along with the thickness of the core. Therefore, the elasticity modulus and density are expressed as follows (Shafiei and Kazemi 2017)

\[ \begin{align*}
E(z) &= E_1 \left( 1 - e_1 \cos \left( \frac{\pi z}{h_c} \right) \right) \\
\rho(z) &= \rho_1 \left( 1 - e_m \cos \left( \frac{\pi z}{h_c} \right) \right)
\end{align*} \]  

(22)

2.4.3 Uniform porosity distribution

For this type of porosity distribution, there is no dependence on \( z \), so Panah et al. (2019)

\[ \begin{align*}
E(z) &= E_1 (1 - e_1), \\
\rho(z) &= \rho_1 (1 - e_1 \chi)
\end{align*} \]  

(23)

where

\[ \chi = \frac{1}{e_1} - \frac{1}{e_1 \sqrt{1 - e_1 - \frac{2}{\pi} + 1}} \]  

(24)

3. Equations derivation

3.1 Hamilton’s principle

To extract the motion equations, Hamilton’s principle is employed. This principle is expressed as (Berghouti et al. 2019)

\[ \delta \int_{t_1}^{t_2} \left[ (W_{\text{ext}} + T) - U \right] \, dt = 0 \]  

(25)

in which \( W_{\text{ext}} \) is the external work, \( U \) is the strain energy and \( T \) denotes the kinetic energy of curved microbeam.

As mentioned before, the current study deals with the behavior of the curved beam in small dimensions. Thus, MCST is employed to analyze the structure at the micro-scale. Based on the MCST, the strain energy can be written as following (Ghorbanpour Arani and Haghighast 2017)

\[ U = \frac{1}{2} \int_{V} \left( \sigma : \varepsilon + m : \chi \right) \, dV \]  

(26)
In this relation, $\sigma$ and $\epsilon$ are the stress and strain tensors, respectively, and $m$ is the deviator part of the couple stress tensor and $\chi$ is the symmetric part of the curvature tensor, which they are defined as follows

$$m_{ij} = 2t^2_m \mu \chi_{ij} \quad (27)$$

$$\chi_{ij} = \frac{1}{2} \left( \Theta_{i,j} + \Theta_{j,i} \right) \quad (28)$$

where $L_m$ is the small scale parameter for the MCST and the Lame’s coefficient is shown by $\mu$. Also, $\Theta$ is the rotation vector which is expressed as

$$\theta = \frac{1}{2} \psi \times \textbf{u} \quad (29)$$

in which $u$ denotes the displacement vector.

Therefore, the expanded form of the strain energy for the under consideration three-layered microbeam is written as follows.

$$U = \frac{1}{2} \int \int \int \left[ \frac{1}{2} \left\{ \sigma_{xx} \frac{\partial \epsilon_{xx}}{\partial x} + 2 \sigma_{xy} \frac{\partial \epsilon_{xy}}{\partial y} + \sigma_{yy} \frac{\partial \epsilon_{yy}}{\partial y} \right\} \right] \text{dV} + \text{2m}_{xy} \dot{\phi} \text{dV} + \frac{1}{2} \int \int \int \left[ \sigma_{xx} \frac{\partial \epsilon_{xx}}{\partial x} + \sigma_{yy} \frac{\partial \epsilon_{yy}}{\partial y} + \sigma_{xy} \frac{\partial \epsilon_{xy}}{\partial y} \right] \text{dV} + \frac{1}{2} \int \int \int \left[ \sigma_{xx} \frac{\partial \epsilon_{xx}}{\partial x} + \sigma_{yy} \frac{\partial \epsilon_{yy}}{\partial y} + \sigma_{xy} \frac{\partial \epsilon_{xy}}{\partial y} \right] \text{dV}$$

$$+ \text{2m}_{xy} \dot{\phi} \text{dV}$$

By employing the variational calculus and integration by part, yields

$$\delta U = \int_A \left[ \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial \epsilon_{xx}}{\partial x} \right) - \frac{1}{2} \frac{\partial}{\partial y} \left( \frac{\partial \epsilon_{yy}}{\partial y} \right) - \frac{1}{2} \frac{\partial}{\partial y} \left( \frac{\partial \epsilon_{xy}}{\partial y} \right) - \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial \epsilon_{xy}}{\partial x} \right) \right] \text{dV}$$

$$\delta U = \int_A \left[ \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial \epsilon_{xx}}{\partial x} \right) - \frac{1}{2} \frac{\partial}{\partial y} \left( \frac{\partial \epsilon_{yy}}{\partial y} \right) - \frac{1}{2} \frac{\partial}{\partial y} \left( \frac{\partial \epsilon_{xy}}{\partial y} \right) - \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial \epsilon_{xy}}{\partial x} \right) \right] \text{dV}$$

The used stress resultants are defined as follows

$$\{ N_{xx}, M_{xx}, H_{xx} \} = \int_{z} \sigma_{xx} \{ 1, z, f(z) \} \text{d}z$$

$$\{ Q_{xx}, R_{xx}, P_{xx}, T_{xx} \} = \int_{z} \sigma_{xx} \{ 1, z, f(z), \frac{d}{dz} f(z) \} \text{d}z$$

$$\{ Y_{1}, Y_{2}, Y_{3}, Y_{4} \} = \int_{z} m_{xy} \{ 1, z, f(z), \frac{d}{dz} f(z) \} \text{d}z$$

$$\{ Y_{5}, Y_{6}, Y_{7} \} = \int_{z} m_{yz} \{ 1, \frac{d}{dz} f(z), \frac{d^2}{dz^2} f(z) \} \text{d}z$$

$$\mathbf{D}_{x} = \int_{z} D_{x} \cos \left( \frac{2\pi}{H_f} \right) \text{d}z$$

$$\mathbf{D}_{z} = \int_{z} D_{z} \sin \left( \frac{2\pi}{H_f} \right) \text{d}z$$

Also, the kinetic energy of the structure can be calculated as follow (Amir et al. 2019)

$$T = \frac{1}{2} \int_{x} \int_{y} \int_{z} \dot{b}(z) \left( \frac{\partial U}{\partial t} + \frac{\partial V}{\partial t} + \frac{\partial W}{\partial t} \right) \text{d}V$$

And in the following it can be derived

$$\delta T = \int_{A} \left[ \left( \begin{array}{c} \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial z^2} u + \frac{\partial^2}{\partial x \partial t^2} w - \frac{\partial^2}{\partial t^2} \phi \frac{\partial^2}{\partial z^2} \delta u + \frac{\partial^2}{\partial x \partial t^2} \frac{\partial^2}{\partial z^2} \delta w \\
- \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial t^2} w - \frac{\partial^2}{\partial x \partial t^2} \frac{\partial^2}{\partial z^2} \delta w + \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial t^2} \phi \frac{\partial^2}{\partial z^2} \delta w \\
\frac{\partial}{\partial x} \frac{\partial}{\partial z} \delta w \\
\frac{\partial}{\partial x} \frac{\partial}{\partial z} \delta \phi \\
\frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial z^2} \delta w \\
\frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial t^2} \phi \frac{\partial^2}{\partial z^2} \delta w \\
\end{array} \right) \right] \text{d}A$$

in which

$$\{ l_0, l_1, l_2, l_3, l_4 \} = \int_{z} \rho(z) \{ 1, z, z^2, f(z), f^2(z) \} \text{d}z$$

The external work in this study consists of two parts: the first due to the elastic foundation and the second due to the mechanical and electrical preloads. So

$$W_{ext} = W_{foundation} + W_{preloads} \quad (36)$$

To determine the work of the elastic foundation, the Pasternak type is selected as a foundation. Therefore, the foundation force can be obtained via the following relation (Sobhy and Zenkour 2018)

$$f_{foundation} = K_{w} w(x, t) - K_{p} \psi^2 w(x, t) \quad (37)$$

where $K_w$ and $K_p$ are the Winkler and shear layer coefficients, respectively.

Consequently, to obtain the work caused by the elastic foundation the following relation may be used (Shahsavari et al. 2019).

$$W_{foundation} = \frac{1}{2} \int_{A} \left[ - f_{foundation} w(x, t) \right] \text{d}A \quad (38)$$

Moreover, the work of electro-mechanical preloads can be determined using the below equation as (Amir et al. 2019).

$$W_{preloads} = \frac{1}{2} \int_{A} \left[ N_{pre} \left( \frac{\partial w(x, t)}{\partial x} \right)^2 \right] \text{d}A \quad (39)$$
\[ \frac{\partial^2 u}{\partial x^2} M_{xx} + \frac{N_{xx}}{R} + \frac{Q_x}{R^2} Y_1 + I_0 \frac{\partial^2}{\partial t^2} u = 0 \]

\[ \frac{\partial^2 w}{\partial x^2} M_{xx} + \frac{N_{xx}}{R} + \frac{Q_x}{R^2} Y_1 - \frac{1}{2R \partial x^2} Y_2 + \frac{1}{2R \partial x} Y_0 - K_w w + K_u \frac{\partial^4}{\partial x^4} w = 0 \]

\[ \frac{\partial^2 \psi}{\partial x} + I_5 \frac{\partial^4 \phi}{\partial x^4 \partial t^2} + I_0 \frac{\partial^2}{\partial t^2} \psi = 0 \]

\[ \frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial x^2} \right) + T_x + \frac{P_z}{R} - \frac{1}{2} \frac{\partial}{\partial x} Y_4 - \frac{1}{2R \partial x} Y_3 + \frac{1}{2} Y_5 \]

\[ \frac{\partial}{\partial x} \left( \frac{\partial^2 \psi}{\partial x^2} \right) - I_5 \frac{\partial^4 \phi}{\partial x^4 \partial t^2} + I_0 \frac{\partial^2}{\partial t^2} \psi = 0 \]
Table 3 Effect of porosity coefficient on the first three natural frequencies (kHz)

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>e_1</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
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<tbody>
<tr>
<td>1st</td>
<td>SSDBT</td>
<td>267.5324</td>
<td>265.5571</td>
<td>263.6130</td>
<td>261.7303</td>
<td>259.9578</td>
<td>258.3781</td>
<td>257.1408</td>
</tr>
<tr>
<td></td>
<td>Tan-SDBT</td>
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<td>266.3304</td>
<td>264.4159</td>
<td>262.5640</td>
<td>260.8232</td>
<td>259.2763</td>
<td>258.0723</td>
</tr>
<tr>
<td>2nd</td>
<td>SSDBT</td>
<td>1073.1119</td>
<td>1064.9662</td>
<td>1056.9274</td>
<td>1049.1127</td>
<td>1041.7131</td>
<td>1035.0542</td>
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<tr>
<td></td>
<td>Tan-SDBT</td>
<td>1082.3880</td>
<td>1074.1830</td>
<td>1066.0804</td>
<td>1058.1966</td>
<td>1050.7211</td>
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<td>1038.5574</td>
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<tr>
<td>3rd</td>
<td>SSDBT</td>
<td>2361.6739</td>
<td>2343.7047</td>
<td>2325.9736</td>
<td>2308.7388</td>
<td>2292.4214</td>
<td>2277.7401</td>
<td>2266.0016</td>
</tr>
<tr>
<td></td>
<td>Tan-SDBT</td>
<td>2395.4826</td>
<td>2377.0738</td>
<td>2358.8689</td>
<td>2341.1203</td>
<td>2324.2416</td>
<td>2308.9433</td>
<td>2296.5241</td>
</tr>
</tbody>
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Table 4 Effect of porosity coefficient on the first three natural frequencies (kHz)

<table>
<thead>
<tr>
<th>Porosity distribution type</th>
<th>Mode No.</th>
<th>e_1</th>
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<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-symmetric</td>
<td>SSDBT</td>
<td>267.5324</td>
<td>265.5571</td>
<td>263.6130</td>
<td>261.7303</td>
<td>259.9578</td>
<td>258.3781</td>
<td>257.1408</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Tan-SDBT</td>
<td>268.2774</td>
<td>266.3304</td>
<td>264.4159</td>
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<td>259.2763</td>
<td>258.0723</td>
<td></td>
</tr>
<tr>
<td>Symmetric</td>
<td>SSDBT</td>
<td>267.5324</td>
<td>266.4189</td>
<td>265.4664</td>
<td>264.7354</td>
<td>264.3153</td>
<td>264.3442</td>
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<td></td>
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<tr>
<td></td>
<td>Tan-SDBT</td>
<td>268.2774</td>
<td>267.2161</td>
<td>266.3214</td>
<td>265.6546</td>
<td>265.3064</td>
<td>265.4171</td>
<td>266.2178</td>
<td></td>
</tr>
<tr>
<td>Uniform</td>
<td>SSDBT</td>
<td>267.5324</td>
<td>265.3053</td>
<td>263.0146</td>
<td>260.6596</td>
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<td>255.7709</td>
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</tr>
<tr>
<td></td>
<td>Tan-SDBT</td>
<td>268.2774</td>
<td>266.0779</td>
<td>263.8182</td>
<td>261.4983</td>
<td>259.1214</td>
<td>256.6971</td>
<td>254.2489</td>
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</tr>
</tbody>
</table>

Fig. 3 Effect of CNTs distribution types on the results

Fig. 4 Central angle variations effect on the fundamental natural frequency

Fig. 5 Effect of CNTs volume fraction for SSDBT and Tan-SDBT

Fig. 6 Effect of layers’ thicknesses and pores distribution on the fundamental natural frequency
Analytical solution for analyzing initial curvature effect on vibrational behavior of PM beams integrated with FGP layers

Fig. 7 Size effect on the fundamental natural frequency for SSDBT and Tan-SDBT

Table 5 Size effect for different volume fractions of CNTs on the natural frequencies (kHz)

<table>
<thead>
<tr>
<th>$V_{CNT}$</th>
<th>$L_m/H$</th>
<th>SSDBT</th>
<th>Tan-SDBT</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.12</td>
<td>0.1</td>
<td>205.8881</td>
<td>213.6967</td>
</tr>
<tr>
<td>0.17</td>
<td>0.2</td>
<td>206.1498</td>
<td>214.0442</td>
</tr>
<tr>
<td>0.28</td>
<td>0.3</td>
<td>230.6485</td>
<td>237.6717</td>
</tr>
<tr>
<td></td>
<td>0.38</td>
<td>231.1783</td>
<td>238.3341</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>273.4149</td>
<td>279.3989</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>274.3596</td>
<td>280.5120</td>
</tr>
</tbody>
</table>

2013, Dan et al. 2016, Ghorbanpour Arani et al. 2018). In this table, the thickness of the beam is considered as 0.2 m, its length is five times more than its thickness, the Poisson’s ratio is 0.3, the elasticity modulus is 75 GPa, and the density of the structure is 2700 kg/m$^3$.

Regarding Table 1, it can be found that the results of the present study are in good convergence with those of literature. Noted that the little difference between the results of the present and previous studies is raised from different selected displacement fields and also different solution methods which lead to this difference. Therefore, the results of the current study are validated and in the following, the effect of the most important parameters on the vibrational behavior of a three-layered curved microbeam is presented.

5.2 Case study

To analyze the effect of the different parameters on the vibrational behavior of the under consideration microbeam, firstly the material properties of the FG porous core and FG CNTRCs face sheets are presented. The material type of the porous core is Tennessee marble with elasticity Young’s modulus of 60 GPa and a density of 2700 kg/m$^3$. Also, its Poisson’s ratio is 0.25. On the other hand, the matrix of the face sheets is from PVDF which has electro-mechanical properties and as stated before, the CNTs are chosen as the reinforcement. Material properties of the Polyvinylidene Fluoride (PVDF) and SWCNTs are presented in Table 2

Fig. 8 Axial wave number effect for different aspect ratio values on the results

Fig. 9 Winkler coefficient of elastic foundation effect on the results

Fig. 10 Effect of shear layer coefficient on the natural frequency of the structure
the results of CNTs are distributed uniformly. Furthermore, the following specifications are used to extract the results: $h_v = 100 \, \mu m$, $h_b = 10 \, \mu m$, $L = 10H$, $L_m = H/2$, $\theta = \pi/6$, $e_1 = 0.3$ and $V_{\text{CNT}}^* = 0.12$.

Table 3 illustrates the effect of porosity coefficient variations on the first three natural frequencies of the curved microbeam. It can be found that by increasing the porosity coefficient which indicates the pores to bulk volume ratio, the stiffness of the structure will be reduced more than its mass, and since the natural frequency is proportional to the square root of stiffness to mass ratio, so the natural frequency decreases too. Also, the results of Table 3 are presented based on both SSDBT and Tan-SDBT and can be seen that there is little difference between them.

Table 4 considers the effect of pores distribution type on the natural frequencies of the structure. The results are presented based on porosity coefficient variation and can be understood that the symmetric type of pores distribution leads to the most values of natural frequencies and the uniform type leads to the least ones. It may be raised due to their functions which are presented in Eqs. (20), (22) and (23).

Fig. 3 presents two effects: the first one is about CNTs distribution type and the second one is about the aspect ratio of the structure. by considering this figure, can be found that the FG V-A type has the maximum natural frequency and the FG A-V has the minimum ones among the considered distribution types. In fact, for FG V-A type, the CNTs are more in surfaces of the structure which enhances its stiffness but in FG A-V type, the CNTs are more about its mid-plane. So, the natural frequencies for this type are less than the others. On the other hand, by increasing the length of the curved microbeam rather than its thickness, the stiffness of the structure will be reduced, and accordingly, the frequencies will be decreased.

Fig. 4 depicts the effect of central angle $\theta$ and CNTs volume fraction values on the results. Based on the findings from this figure, increasing the central angle which means a reduction in curvature radius, leads the results to reduce, while increasing the volume fraction of CNTs leads them to enhance. CNTs’ stiffness is much more than the matrix, consequently increasing their volume fraction than matrix, which will increase the face sheets stiffness, and in the following the stiffness of the structure will be increased which means an increase in the frequencies.

In Fig. 5, the effect of slenderness and the amount of CNTs is considered. Regarding this figure, can be seen that as the CNTs’ amount increases with respect to the matrix phase, the natural frequencies increase, too. Also, as the beam becomes longer, it vibrates more.

The effect of thicknesses ratio is studied in Fig. 6. It is seen that by keeping the total thickness constant, increasing the core’s thickness means a reduction to that of the face sheets. Since the face sheets, stiffness is much more than the core, so the reduction in their thickness will be ended to the reduction in stiffness of the whole of the structure. Therefore, the natural frequencies will be reduced. Also, the effect of pores distribution types can be seen in this figure which confirms the previous findings.

Size effect for two cases namely two CNTs distribution types and different values of CNTs volume fractions are considered in Fig. 7 and Table 5, respectively. Regarding these items, increasing the small-scale parameters $L_m$, the stiffness of the structure will be enhanced rapidly, and following it, the fundamental natural frequency of the structure will be increased.

Axial wave number increasing effect which shows the vibrational mode number is shown in Fig. 8 and as can be expected, its increasing leads the frequencies to enhance. This figure is plotted for different aspect ratios which ensure the previous findings.

The presented results up to here were in absence of an elastic foundation. But, now the effect of adding elastic
foundation and variation of its coefficients will be discussed. Fig. 9 shows the effect of the spring or Winkler coefficient increasing on the results. Increasing $K_w$ enhances the stiffness of the structure and accordingly, the frequency will be reduced.

A similar effect but for $K_G$ is considered in Fig. 10 and the variations of natural frequency are like that of $K_w$.

Fig. 11 shows the effect of both spring and shear layer coefficients simultaneously and can be found shear layer coefficient has more effect on the results than that of spring.

Also, Fig. 12 considered three different cases, namely without elastic foundation, with the Winkler elastic foundation which only consists of springs and Pasternak elastic foundation which consists of both springs and shear layer. It is understood that adding the elastic foundation enhances the stiffness and natural frequencies of the structure and adding the shear layer which converts the Winkler type to Pasternak one, has a similar effect on the vibrational behavior of the curved microbeam.

Finally, the effect of electro-mechanical preloads is considered and listed in Table 6. According to this table, the positive mechanical preload which means tensile load increases the stability of the structure, and following it, as the tensile load becomes more, the frequency will be enhanced. But for negative preload which means compressive load, makes the structure unstable and as the compressive load becomes more, the frequency will be reduced. It is noticeable that this effect is not significant.

Moreover, about electrical preload, the positive applied potential leads the frequencies to be reduced, while the negative potential leads them to enhance due to increasing the stability of the structure.

<table>
<thead>
<tr>
<th>$V_{CNT}^*$</th>
<th>$P_0$</th>
<th>$\phi_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SSDBT</td>
<td>Tan-SDBT</td>
</tr>
<tr>
<td>-100</td>
<td>261.6676</td>
<td>262.5015</td>
</tr>
<tr>
<td>0.12</td>
<td>261.7832</td>
<td>262.6167</td>
</tr>
<tr>
<td>100</td>
<td>261.8987</td>
<td>262.7318</td>
</tr>
<tr>
<td>-100</td>
<td>281.9051</td>
<td>283.0264</td>
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<tr>
<td>0.17</td>
<td>282.0126</td>
<td>283.1335</td>
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<tr>
<td>100</td>
<td>282.1200</td>
<td>283.2404</td>
</tr>
<tr>
<td>-100</td>
<td>318.0766</td>
<td>319.6794</td>
</tr>
<tr>
<td>0.28</td>
<td>318.1722</td>
<td>319.7745</td>
</tr>
</tbody>
</table>

Table 6 Influence of different values of electro-mechanical preloads on the results (kHz)

6. Conclusions

In the current study, the free vibrational behavior of a three-layered curved microbeam is considered based on the two higher-order and trigonometric shear deformation theories namely SSDBT and Tan-SDBT. The core of the structure is made from FG porous mat and the face sheets are from FG CNTRCs which are exposed to the electric field. The three layers’ mechanical properties are varied through their thickness following to given functions. Hamilton’s principle is used to extract the motion equations and the MCST is used to capture the size effect. To solve the motion equations analytically, Navier’s method is employed for a simply supported ends case and by validating the results for the simpler state with previous works, effects of different parameters on the behavior of the structure are considered and the following items can be concluded:

- Increasing the porosity coefficient leads the natural frequencies to reduce.
- Among the three different types of porosity distributions, symmetric and uniform types have the most and the least values of natural frequencies, respectively.
- FG V-A and FG A-V patterns of CNTs distribution have the maximum and minimum values of the results, respectively.
- Increasing the volume fraction of CNTs in the face sheets enhances the stiffness and accordingly the natural frequencies of the structure.
- Increasing the small scale parameter, lead the frequencies to enhance.
- Increasing the central angle of the curved microbeam causes a reduction in the natural frequencies.
- By keeping the thickness constant, as the length of the structure increases, the natural frequency will be reduced.
- Increasing the core’s thickness and reducing the face sheets’ ones, lead the frequency to reduce.
- Adding the elastic foundation to the structure increases its stiffness and frequencies.
- Frequencies of the Pasternak type of elastic foundation are higher than those of Winkler.
- Increasing the tensile mechanical preload, increases the natural frequencies, while it is vice versa about the compressive load.
- Although increasing the positive applied electrical preload reduces the natural frequencies, but the negative one enhances it.

The obtained results based on Tan-SDBT are generally more than those of SSDBT.
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References


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The non-zero components of stiffness and mass matrices introduced in Eq. (47) are defined as follow.

\[
K_{11} = \frac{Q_{550}}{R^2} + \frac{G_0}{R} + Q_{110}a^2 + A_{14}a^2 + \frac{L_{m}^2 H_2a^2}{4 R^2} + \frac{L_{m}^2 T_1a^2}{4 R^2} - \frac{K_1a}{R^2} - \frac{Q_{110}a}{R^2} - \frac{Q_{555}a}{R^2} - G_1a
\]

\[
K_{12} = -Q_{111}a^2 - A_{13}a^2 - \frac{L_{m}^2 H_2a^2}{4 R^2} - \frac{L_{m}^2 H_3a^2}{4 R^2} - \frac{L_{m}^2 T_2a^2}{4 R^2} + \frac{L_{m}^2 H_4a^2}{4 R^2} + \frac{L_{m}^2 T_1a^2}{4 R^2} - \frac{A_{10}a}{R^2} - \frac{Q_{110}a}{R^2} - \frac{Q_{555}a}{R^2} - G_1a
\]

\[
K_{13} = \frac{Q_{556}}{R} + \frac{Q_{553}}{R^2} + \frac{G_6}{R} + \frac{Q_{1113}a^2 + A_{13}a^2 + \frac{L_{m}^2 H_2a^2}{4 R^2} + \frac{L_{m}^2 T_1a^2}{4 R^2}}{4 R^2} - \frac{A_{10}a}{R^2} - \frac{Q_{110}a}{R^2} - \frac{Q_{555}a}{R^2} - G_1a
\]

\[
K_{14} = -E_{310}a - \frac{E_{150}a}{R}, \quad K_{24} = \frac{E_{310}}{R} + E_{311}a^2 + \frac{E_{151}a^2}{R}
\]

\[
K_{31} = \frac{G_3}{R} + \frac{Q_{556}}{R^2} + \frac{Q_{553}}{R^2} + \frac{L_{m}^2 H_2a^2}{4 R^2} + \frac{L_{m}^2 T_4a^2}{4 R^2} + \frac{H_2L_2a^2}{4 R^2} + \frac{L_{m}^2 T_1a^2}{4 R^2} - \frac{G_3a}{R} - \frac{1}{2} L_{m}^2 H_2a^2 - \frac{Q_{555}a}{R^2} - \frac{Q_{555}a}{R^2} - \frac{G_3a}{R} - \frac{1}{2} L_{m}^2 T_2a^2
\]

\[
K_{41} = -E_{310}a - \frac{E_{150}a}{R}, \quad K_{31} = -E_{153}a - R - E_{151}a
\]

\[
K_{42} = -E_{310} + E_{311}a^2 + \frac{E_{151}a^2}{R}, \quad K_{44} = -S_{110}a^2 - S_{330}, \quad M_{11} = - \Lambda_{1}, \quad M_{12} = M_{21} = \Lambda_{1}, \quad M_{13} = M_{31} = - \Lambda_{1}, \quad M_{22} = - \Lambda_{2}^2 - \Lambda_{1}, \quad M_{23} = M_{32} = \Lambda_{2}, \quad M_{33} = - \Lambda_{2}
\]

in which

\[
Q_{11i} = \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \int_{-h_{ho}}^{h_{ho}} Q_{11i}z^i dz, \quad i = 0, 1, 2
\]

\[
Q_{113} = \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \int_{-h_{ho}}^{h_{ho}} Q_{113}f(z) dz
\]

\[
Q_{114} = \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \int_{-h_{ho}}^{h_{ho}} Q_{114}f(z) dz
\]

\[
Q_{115} = \int_{-\frac{h_c}{2}}^{\frac{h_c}{2}} \int_{-h_{ho}}^{h_{ho}} Q_{115}z^i dz, \quad i = 0, 1, 2
\]
\[ Q_{553} = \int_{\frac{h_c}{2}}^{\frac{h_c}{2} - \epsilon_b} Q_{555} f(z) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + \epsilon_b} Q_{555} f(z) dz \]

\[ Q_{554} = \int_{\frac{h_c}{2}}^{\frac{h_c}{2} - \epsilon_b} Q_{555} f^2(z) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + \epsilon_b} Q_{555} f^2(z) dz \]

\[ Q_{555} = \int_{\frac{h_c}{2}}^{\frac{h_c}{2} - \epsilon_b} Q_{555} z f(z) dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + \epsilon_b} Q_{555} z f(z) dz \]

\[ Q_{556} = \int_{\frac{h_c}{2}}^{\frac{h_c}{2} - \epsilon_b} Q_{555} \frac{df(z)}{dz} dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + \epsilon_b} Q_{555} \frac{df(z)}{dz} dz \]

\[ Q_{557} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} Q_{555} z f(z) dz + \int_{\frac{h_c}{2} + \epsilon_b}^{\frac{h_c}{2}} Q_{555} z f(z) dz \]

\[ Q_{558} = \int_{\frac{h_c}{2}}^{\frac{h_c}{2} - \epsilon_b} Q_{555} \frac{df(z)}{dz} dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + \epsilon_b} Q_{555} \frac{df(z)}{dz} dz \]

\[ Q_{559} = \int_{\frac{h_c}{2}}^{\frac{h_c}{2} - \epsilon_b} Q_{555} \left( \frac{df(z)}{dz} \right)^2 dz + \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + \epsilon_b} Q_{555} \left( \frac{df(z)}{dz} \right)^2 dz \]

\[ A_{15} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} A(z) z \frac{df(z)}{dz} dz, \quad i = 0, 1, 2, \quad A_{13} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} A(z) f(z) dz, \quad A_{14} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} A(z) f^2(z) dz \]

\[ A_{15} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} A(z) z f(z) dz, \quad G_0 = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} G(z) z \frac{df(z)}{dz} dz, \quad i = 0, 1, 2, \quad G_3 = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} G(z) f(z) dz \]

\[ G_4 = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} G(z) f^2(z) dz, \quad G_5 = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} G(z) z f(z) dz, \quad G_6 = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} G(z) \frac{df(z)}{dz} dz \]

\[ E_{310} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} \frac{\pi}{h_b} e^{31} \sin \left( \frac{\pi z_b}{h_b} \right) dz + \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2} + \epsilon_b} \frac{\pi}{h_t} e^{31} \sin \left( \frac{\pi z_t}{h_t} \right) dz \]

\[ E_{311} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} \frac{\pi}{h_b} e^{31} \sin \left( \frac{\pi z_b}{h_b} \right) zdz + \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2} + \epsilon_b} \frac{\pi}{h_t} e^{31} \sin \left( \frac{\pi z_t}{h_t} \right) zdz \]

\[ E_{313} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} \frac{\pi}{h_b} e^{31} \sin \left( \frac{\pi z_b}{h_b} \right) f(z) dz + \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2} + \epsilon_b} \frac{\pi}{h_t} e^{31} \sin \left( \frac{\pi z_t}{h_t} \right) f(z) dz \]

\[ E_{150} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} e^{15} \cos \left( \frac{\pi z_b}{h_b} \right) dz + \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2} + \epsilon_b} e^{15} \cos \left( \frac{\pi z_t}{h_t} \right) dz \]

\[ E_{151} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} e^{15} \cos \left( \frac{\pi z_b}{h_b} \right) zdz + \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2} + \epsilon_b} e^{15} \cos \left( \frac{\pi z_t}{h_t} \right) zdz \]

\[ E_{153} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} e^{15} \cos \left( \frac{\pi z_b}{h_b} \right) f(z) dz + \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2} + \epsilon_b} e^{15} \cos \left( \frac{\pi z_t}{h_t} \right) f(z) dz \]

\[ E_{156} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} e^{15} \cos \left( \frac{\pi z_b}{h_b} \right) \left( \frac{df(z)}{dz} \right)^2 dz + \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2} + \epsilon_b} e^{15} \cos \left( \frac{\pi z_t}{h_t} \right) \left( \frac{df(z)}{dz} \right)^2 dz \]

\[ E_{159} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} e^{15} \cos \left( \frac{\pi z_b}{h_b} \right) \left( \frac{df(z)}{dz} \right)^2 dz + \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2} + \epsilon_b} e^{15} \cos \left( \frac{\pi z_t}{h_t} \right) \left( \frac{df(z)}{dz} \right)^2 dz \]

\[ S_{110} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} S_{11} \cos^2 \left( \frac{\pi z_b}{h_b} \right) dz + \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2} + \epsilon_b} S_{11} \cos^2 \left( \frac{\pi z_t}{h_t} \right) dz \]

\[ S_{330} = \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2}} S_{33} \left( \frac{\pi}{h_b} \right)^2 \sin^2 \left( \frac{\pi z_b}{h_b} \right) dz + \int_{\frac{h_c}{2} - \epsilon_b}^{\frac{h_c}{2} + \epsilon_b} S_{33} \left( \frac{\pi}{h_t} \right)^2 \sin^2 \left( \frac{\pi z_t}{h_t} \right) dz \]
Analytical solution for analyzing initial curvature effect on vibrational behavior of PM beams integrated with FGP layers...