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Theoretical analysis of composite beams under uniformly distributed load

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Abstract. The bending problem of a functionally graded cantilever beam subjected to uniformly distributed load is investigated. The material properties of the functionally graded beam are assumed to vary continuously through the thickness, according to a power-law distribution of the volume fraction of the constituents. First, the partial differential equation, which is satisfied by the stress functions for the axisymmetric deformation problem is derived. Then, stress functions are obtained by proper manipulation. A practical example is presented to show the application of the method.

Keywords: functionally graded beam; uniformly distributed load; elastic properties; analytical solution

1. Introduction

Functionally graded materials are a kind of material possessing properties that vary gradually with respect to the spatial coordinates. The material properties can be designed so as to improve its strength, toughness, high temperature withstanding ability, etc. Compared with traditional laminated composite structures, structures made of FGM have no obvious interfaces of material property, so that the phenomena of stress concentration can be eliminated or weakened greatly. Thus, FGMs have broad potential applications in aeronautics/astronautics manufacturing industry, nuclear power plant, etc.

Beams are one of the important structural members used in many engineering applications, such as in aerospace, automotive and ocean engineering. Studies of beams may be divided into two groups: in terms of the materials from which they are made, and in terms of the boundary conditions and solution method in the plate problem analyzed. Isotropic, composite and functionally graded beams are examples of materials used in beams. Attentions have always been paid to the elasticity solutions for plane beams by scientists and engineers. Exact and analytical elasticity solutions for homogeneous isotropic beams can be obtained via Airy stress function, as shown in Timoshenko and Goodier (1970). Lekhnitskii (1968), using the stress function method, further obtained a series of solutions for plane anisotropic beams, including the one for beams subjected to simple tension, pure shear, and pure bending, the one for cantilever beams acted by a concentrated shear force at the tip, the one for uniformly loaded cantilever beams and simply supported beams, and the one for linearly loaded cantilever beams and simply supported beams. Silverman (1964) presented a general method to obtain stress function for orthotropic beams; the bending problems of cantilever beams subjected to a terminal shear force and cantilever beams subjected to uniform load and linearly distributed load were studied. Recently, Ding (2007) derived an elasticity solution for a fixed–fixed

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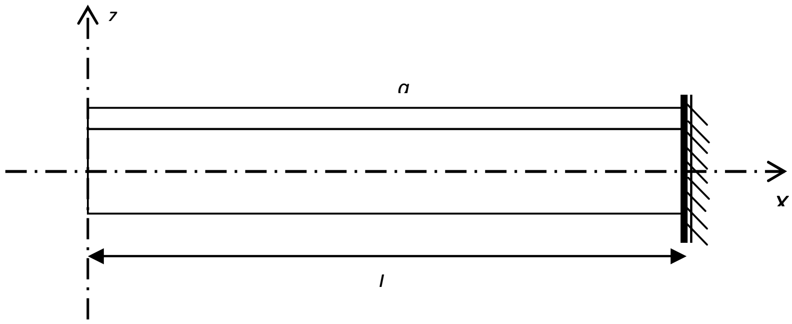

Fig. 1 A cantilever beam under a uniform load and the coordinates

plane isotropic beam subjected to uniform load with the aid of Airy stress function; the correctness of the solution was confirmed through comparison with the numerical solution of Ahmed *et al.*. (1996). However, the literature on the response of functionally graded beam (FGB) to mechanical and other loadings are limited. Functionally graded materials (FGMs) are composite materials in which the mechanical properties vary smoothly and continuously in preferred directions. Macroscopically, FG beams are assumed to be isotropic materials. Therefore the same boundary conditions exist for FG and isotropic beams.

Shi and his coworkers studied the response of FGPM beams (Shi and Chen 2004, Zhang and Shi 2003). But in their analysis, only one or two material parameters were assumed to vary in the form of finite power series along the thickness direction while other parameters kept constant. Sankar and his coworkers (Sankar and Taeng 2002, Venkataraman and Sankar 2003) developed analytical methods for the thermomechanical and contact analysis of FGM beams and also for sandwich beams with FGM cores (Tlidji *et al.* 2014). In their studies the thermomechanical properties of the FGM were all assumed to vary through the thickness in an exponential fashion. Zhu and Sankar (2004) and solved the two-dimensional elasticity equations for a FGM beam subjected to transverse loads by means of combined Fourier series-Galerkin method, in which the variation of the Young's modulus through the thickness was given by a polynomial in the thickness-coordinate and Poisson's ratio was assumed to be constant. 2D analytical solutions for plates and beams have also been presented by Huang (2007), Hassaine daouadji *et al.* (2015), Bellifa *et al.* (2016), Khelifa *et al.* (2015), Bennoun *et al.* (2016) , Habali *et al.* (2014), Belabed *et al.* (2014) and Bourada *et al.* (2015).

In this paper, the stress function approach is employed to study the problem of a functionally graded cantilever beam subjected to uniformly distributed load. The effect of material in-homogeneity parameter on the displacement and stress field in the beam is presented and conclusions are then drawn.

2. Problem description and basic equations

Consider a cantilever FGM beam of uniform thickness h as shown in Fig. 1. A Cartesian coordinate system is introduced with the upper and lower surfaces of the beam lying in the plane $\boldsymbol{z} = \boldsymbol{h}/2$ and $\boldsymbol{z}=-\boldsymbol{h}/2$. The lengths of the edges of the beam in *x* and *y*-direction are denoted by $\boldsymbol{L}$ and $\boldsymbol{b}$, respectively. The beam is assumed to be in a state of plane stress normal to the x–z plane, and is subjected to a uniformly distributed load $\boldsymbol{q}$ on its upper surface. The material properties vary in the thickness direction.

In the absence of body forces the equilibrium equations are given as

$$\frac{\partial \boldsymbol{\sigma}_{\boldsymbol{x}}}{\partial \boldsymbol{x}} + \frac{\partial \tau_{\boldsymbol{xz}}}{\partial \boldsymbol{z}} = 0, \qquad \frac{\partial \tau_{\boldsymbol{xz}}}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{\sigma}_{\boldsymbol{z}}}{\partial \boldsymbol{z}} = 0 \tag{1}$$

where σ_x, σ_z, τ_{xz} are stress components.

The relationships between strains and displacements are

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{2}$$

where u, w are displacement components, ε_x, ε_z, γ_{xz} are strain components that should satisfy the following strain compatibility equation

$$\frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} - \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} = 0 \tag{3}$$

The constitutive relations of orthotropic FGM are

$$\varepsilon_x = S_{11}\sigma_x + S_{13}\sigma_z, \quad \varepsilon_z = S_{13}\sigma_x + S_{33}\sigma_z, \quad \gamma_{xz} = S_{44}\tau_{xz} \tag{4}$$

where S_{11}, S_{33}, S_{13}, S_{44} are the elastic compliance parameters given by

$$S_{11} = S_{33} = \frac{1}{E(z)}, \quad S_{13} = -\frac{\nu}{E(z)}, \quad S_{44} = \frac{2(1+\nu)}{E(z)} \tag{5}$$

The boundary conditions of elasticity at the upper and lower surfaces are

$$\sigma_z(x,-h/2) = -q, \quad \sigma_z(x,+h/2) = 0, \quad \tau_{xz}(x,\pm h/2) = 0. \tag{6}$$

The boundary conditions at the left end (free) of the FGM beam are

$$N_0 = 0, \quad M_0 = 0, \quad Q_0 = 0 \tag{7}$$

where N_0, M_0 and Q_0 denote the axial force, moment and shear force at x=0. The boundary conditions for the fixed end at the right end (fixed) of the beam are taken as

$$u = w = 0, \quad \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = L, \quad z = 0 \tag{8}$$

3. General solution

In order to obtain a general solution to Eqs. (1)-(8), Airy stress function $\Phi(x,z)$ is introduced such that

$$\sigma_x = \frac{\partial^2 \Phi}{\partial z^2}, \quad \sigma_z = \frac{\partial^2 \Phi}{\partial x^2}, \quad \tau_{xz} = -\frac{\partial^2 \Phi}{\partial x \partial z}. \tag{9}$$

So that Eq. (1) is satisfied automatically. We then assume that

$$\Phi(x,z) = \Phi_0(z) + x\Phi_1(z) + x^2\Phi_2(z), \tag{10}$$

where $\Phi_0(z)$, $\Phi_1(z)$ and $\Phi_2(z)$ are unknown functions to be determined. Substitution of Eq. (10) into Eq. (9) gives

$$\sigma_x = \Phi_0'' + x\Phi_1'' + x^2\Phi_2'' \tag{11a}$$

$$\sigma_z = 2\Phi_2, \tag{11b}$$

$$\tau_{xz} = -\left(\Phi_1' + 2x\Phi_2'\right), \tag{11c}$$

Substitution of Eq. (11) into Eq. (4), and then into Eq. (3), gives rise to

$$\left(\boldsymbol{S}_{11}\Phi_2^{''}\right)^{''}=0 \tag{12a}$$

$$\left(\boldsymbol{S}_{11}\Phi_1^{''}\right)^{''}=0 \tag{12b}$$

$$\left(S_{11}\Phi_0^{''}+2S_{13}\Phi_2\right)^{''}+(2S_{44}\Phi_2^{'})^{'}+2S_{13}\Phi_2^{''}=0. \tag{12c}$$

Integration of Eq. (12a) yields

$$\Phi_2^{''}=\boldsymbol{a}_1\boldsymbol{A}_1+\boldsymbol{a}_2\boldsymbol{A}_2 \tag{13a}$$

$$\Phi_2^{'}=\boldsymbol{a}_1\boldsymbol{A}_1^0+\boldsymbol{a}_2\boldsymbol{A}_2^0+\boldsymbol{a}_3 \tag{13b}$$

$$\Phi_2=\boldsymbol{a}_1\boldsymbol{A}_1^1+\boldsymbol{a}_2\boldsymbol{A}_2^1+\boldsymbol{a}_3\boldsymbol{z}+\boldsymbol{a}_9 \tag{13c}$$

where and hereafter $\boldsymbol{a}_i$ ($\boldsymbol{i}$=1,2,…) are integral constants, and

$$\boldsymbol{A}_1=\frac{\boldsymbol{z}}{\boldsymbol{S}_{11}},\quad \boldsymbol{A}_2=\frac{1}{\boldsymbol{S}_{11}},\quad \boldsymbol{A}_i^{\boldsymbol{n}}(\boldsymbol{z})=\frac{1}{\boldsymbol{n}!}\int_{-\boldsymbol{h}/2}^{\boldsymbol{z}}(\boldsymbol{z}-\xi)^{\boldsymbol{n}}\boldsymbol{A}_i d\xi \quad (\boldsymbol{i}=1,2;\boldsymbol{n}=0,1) \tag{14}$$

Integration of Eq. (12b) yields

$$\Phi_1^{''}=\boldsymbol{a}_4\boldsymbol{B}_4+\boldsymbol{a}_5\boldsymbol{B}_5 \tag{15a}$$

$$\Phi_1^{'}=\boldsymbol{a}_4\boldsymbol{B}_4^0+\boldsymbol{a}_5\boldsymbol{B}_5^0+\boldsymbol{a}_6 \tag{15b}$$

where

$$\boldsymbol{B}_4=\boldsymbol{A}_1,\quad \boldsymbol{B}_5=\boldsymbol{A}_2,\quad \boldsymbol{B}_i^{\boldsymbol{n}}(\boldsymbol{z})=\frac{1}{\boldsymbol{n}!}\int_{-\boldsymbol{h}/2}^{\boldsymbol{z}}(\boldsymbol{z}-\xi)^{\boldsymbol{n}}\boldsymbol{B}_i d\xi \quad (\boldsymbol{i}=4,5;\boldsymbol{n}=0,1) \tag{16}$$

Substituting Eqs. (13), (15) into Eq. (12c), and performing integration twice, we obtain

$$\Phi_0^{''}=\sum_{\boldsymbol{i}=1}^{9}\boldsymbol{a}_i\boldsymbol{C}_i \tag{17}$$

where

$$C_1=-\frac{2}{S_{11}}\left[\int_{-h/2}^{z}(z-\xi)S_{13}A_1 d\xi+\int_{-h/2}^{z}S_{44}A_1^0 d\xi+S_{13}A_1^1\right], \tag{18a}$$

$$C_2=-\frac{2}{S_{11}}\left[\int_{-h/2}^{z}(z-\xi)S_{13}A_2 d\xi+\int_{-h/2}^{z}S_{44}A_2^0 d\xi+S_{13}A_2^1\right], \tag{18b}$$

$$C_3=-\frac{2}{S_{11}}[S_{13}z+\int_{-h/2}^{z}S_{44}d\xi], \tag{18c}$$

$$C_4 = C_5 = C_6 = 0, \tag{18d}$$

$$C_7 = A_1\,,\;\; C_8 = A_2\,,\;\; C_9 = -\frac{2S_{13}}{S_{11}} \tag{18e}$$

and

$$C_i^n(z) = \frac{1}{n!}\int_{-h/2}^{z}(z-\xi)^n C_i d\xi \qquad (i = 1,2,....,9;\, n = 0,1) \tag{19}$$

Axial force, Bending moment, Shear force and Displacements

The axial force N_0, bending moment M_0 and shearing force Q_0 at the left end of the beam can be obtained by integrating Eqs. (11a) and (11c) according to

$$N_0 = \int_{-h/2}^{h/2}\sigma_x(0,z)dz = \sum_{i=1}^{9} a_i C_i^0(h/2), \tag{20a}$$

$$M_0 = \int_{-h/2}^{h/2}\sigma_x(0,z)zdz = \frac{h}{2}\sum_{i=1}^{9} a_i C_i^0(h/2) - \sum_{i=1}^{9} a_i C_i^1(h/2), \tag{20b}$$

$$Q_0 = \int_{-h/2}^{h/2}\tau_{xz}(0,z)dz = -\big(a_4 B_4^1(h/2) + a_5 B_5^1(h/2)\big) - a_6 h. \tag{20c}$$

Substituting Eqs. (13), (15) and (17) into Eq. (11), then into Eq. (4), and finally performing the integration, we could obtain the displacement components as follows

$$u = x\left(S_{11}\Phi_0'' + 2S_{13}\Phi_2\right) + \frac{x^2}{2}S_{11}\Phi_1'' + \frac{x^3}{3}S_{11}\Phi_2'' - \int_{-h/2}^{z} S_{44}\Phi_1' d\xi - \int_{-h/2}^{z}(z-\xi)S_{13}\Phi_1'' d\xi - \omega\, z + u_0, \tag{21a}$$

$$w = \int_{-h/2}^{z}\left(s_{13}\Phi_0'' + 2S_{33}\Phi_2\right)d\xi - \frac{a_1}{12}x^4 - \frac{a_4}{6}x^3 + \left(\int_{-h/2}^{z} S_{13}\Phi_2'' d\xi - \frac{a_7}{2}\right)x^2 + \left(\int_{-h/2}^{z} S_{13}\Phi_1'' d\xi + \omega\right)x + w_0, \tag{21b}$$

where u_0, w_0 and ω are integral constants.

Substitution of Eqs. (13b), (13c) and (15b) into Eqs. (11b) and (11c), and then into Eq. (6), yields

$$a_3 = 0 \quad , \quad a_6 = 0 \quad , \quad a_9 = -q/2 \quad , \quad \sum_{i=}^{2} a_i A_i^0(h/2) = 0 \quad , \quad 2\sum_{i=}^{2} a_i A_i^1(h/2) - q = 0 \quad , \tag{21a}$$

$$a_4 B_4^0(h/2) + a_5 B_5^0(h/2) = 0.$$

Substitution of Eq. (20) into Eq. (7) gives

$$\sum_{i=1}^{9} a_i C_i^0(h/2) = 0\,, \quad \sum_{i=1}^{9} a_i C_i^1(h/2) = 0\,, \quad a_4 B_4^1(h/2) + a_5 B_5^1(h/2) = 0. \tag{22}$$

The unknown constants a_i can be obtained from Eqs. (22) and (23). Thus, all the undetermined constants are fixed, and the stress function Φ is determined completely. The stress components are then obtained from Eq. (11).

Substitution of Eqs. (21) into Eq. (8) yields three equations, from which u_0, v_0 and ω can be obtained. Thus, the displacement components u and v are determined.

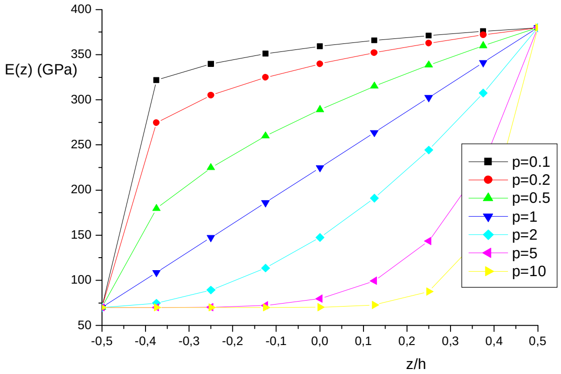

Fig. 2 The variation of Young뭪 modulus in a PFGM beam

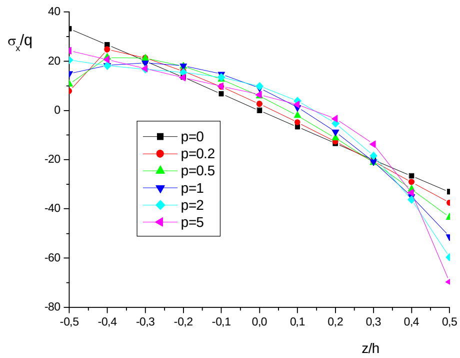

Fig. 3 Dimensionless normal stress(σ_x)/q of PFGM cantilever beam under a uniform load at x=l/3

4. Numerical results and discussion:

In this section, numerical study of a FGM cantilever beam (l = 1m, height h = 0.1 m) subjected to a linearly distributed load q_0=10^6 N/m. The analysis is performed for pure materials and different values of material parameter ***p***, for aluminum–alumina FGM. The Young's modulus and Poisson's ratio (Sallai and al. 2009) for aluminum are: 70 GPa and 0.3 and for alumina: 380 GPa and 0.3, respectively on the upper surface of the beam will be made based on the above solution. We assume the volume fraction of the P-FGM is assumed to obey a power law function

$$g(z) = \left(\frac{z}{h} + \frac{1}{2}\right)^{p} \tag{24}$$

where ***p*** is the material parameter that dictates the material variation profile through the thickness and ***h*** is the thickness of the beam. Once the local volume fraction *g(z)* has been defined, the material properties of a P-FGM can be determined by the rule of mixture

$$E(z) = g(z)E_1 + (1 - g(z))E_2 \tag{25}$$

where $\boldsymbol{E}_1$ and $\boldsymbol{E_2}$ are the Young modules of the lowest (***z* = *h*/2**) and top surfaces (***z* = - *h*/2**) of the FGM beam, respectively. The variation of Young's modulus in the thickness direction of the P-FGM beam is

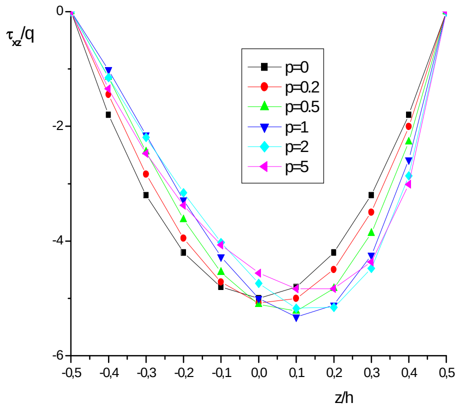

Fig. 4 Dimensionless tangential stress (τ_{xz})/q of PFGM cantilever beam under a uniform load at x=L/3

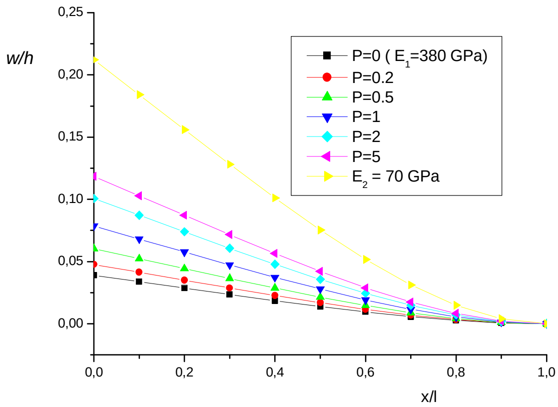

Fig. 5 Deflection of the centerline for different *P* of PFGM cantilever beam under a uniform load

depicted in Fig. 2, which shows that the Young's modulus changes rapidly near the lowest surface for $\boldsymbol{p > 1}$, and increases quickly near the top surface for $\boldsymbol{p < 1}$.

In present study, we take the graded index for different values of *p* in P-FGM. The material properties at $z = z_0 = 0$ are given as

$$s_{11}^0 = 5.41\ x10^{-12}\ /\ Pa\ , \quad s_{13}^0 = -1.51\ x10^{-12}\ /\ Pa\ , \quad s_{33}^0 = 9.52\ x10^{-11}\ /\ Pa\ , \quad s_{44}^0 = 1.37\ x10^{-10}\ /\ Pa\ . \tag{26}$$

Fig 3 show the dimensionless normal stress σ_x at $x=l/3$; Fig 4 show the dimensionless tangential stress τ_{xz} at $x=l/3$; Fig 5 show the variation of deflection curves of PFGM cantilever beam at $z=0$. It is observed that the deflection *w* varies linearly with coordinate *x* at $z=0$. The distribution of stress σ_x is nonlinear along the thickness direction for functionally graded materials ($p\neq 0$) different from the uniformly distribution for a homogeneous material ($p=0$). The stresses, σ_z and τ_{xz}, are much smaller compared to σ_x. It is worthy to notice that σ_z varies nonlinearly from zero at the lower surface ($z = h/2$) to the given pressure at the upper surface ($z = - h/2$) and the stress τ_{xz} vanishes at both upper and lower surfaces, from which the given boundary conditions are satisfied and the correctness of the present solution is validated. The stress and displacement field of the beam is greatly influenced by graded index *p*, which means that the beam can be optimally design according to given working conditions by tailoring the graded material properties.

5. Conclusion

A plane elasticity solution is developed for a functionally graded beam by means of the semi-inverse method. The mechanical properties of the material are assumed to have the same dependence on the thickness-coordinate and a general two-dimensional solution is obtained for a cantilever functionally graded beam subjected to normal and shear tractions on the upper and lower surfaces as well as applied concentrated forces and couple at the free end. The obtained solution is valid for arbitrary graded variations of the material distribution, so it could serve as a basis for establishing simplified FGM theories or as a benchmark result to assess other approximate methodologies.

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