Advances in Concrete Construction, *Vol. 6, No. 6 (2018) 585-610*
DOI: https://doi.org/10.12989/acc.2018.6.6.585

Agglomerated SiO_2 nanoparticles reinforced-concrete foundations based on higher order shear deformation theory: Vibration analysis

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(Received September 14, 2018, Revised October 24, 2018, Accepted October 26, 2018)

Abstract. In this study, vibration analysis of a concrete foundation-reinforced by SiO_2 nanoparticles resting on soil bed is investigated. The soil medium is simulated with spring constants. Furthermore, the Mori-Tanaka low is used for obtaining the material properties of nano-composite structure and considering agglomeration effects. Using third order shear deformation theory or Reddy theory, the total potential energy of system is calculated and by means of the Hamilton's principle, the coupled motion equations are obtained. Also, based an analytical method, the frequency of system is calculated. The effects of volume percent and agglomeration of SiO_2 nanoparticles, soil medium and geometrical parameters of structure are shown on the frequency of system. Results show that with increasing the volume percent of SiO_2 nanoparticles, the frequency of structure is increased.

Keywords: vibration of concrete foundation; agglomeration; SiO_2 nanoparticles; soil medium; analytical method

1. Introduction

Free vibrational analysis of concrete foundations resting on soil bed is of great importance for the design of many engineering problems such as footing of buildings, pavement of roads and bases of machines. It is due to the vibrational effect on the performance of sensitive equipment located on foundations. Engineers need to know natural frequencies and corresponding mode shapes of foundations so that they can design system appropriately. Also, as concrete is most usable material in the construction of foundations, it's been required to improve its quality for reducing vibrations. Nowadays, nanotechnology offers the possibility of great advances in construction materials. So, free vibrations can be reduced by improving the properties of concrete foundations with adding nano material in order to increase its stiffness. On the other hand, for the accurate vibrational analysis of thick plates such as foundations, it is appropriate to use the high-order shear deformation theories. Because the results evaluated by using classical thin plate theory may not be reliable especially as the plate gets thicker. Therefore, free vibration analysis of

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http://www.techno-press.org/?journal=acc&subpage=7 ISSN: 2287-5301 (Print), 2287-531X (Online)

concrete foundations reinforced with SiO_2 nanoparticles using shear deformation theory is done on this topic. Also, Winker model is used for the simulation of soil bed because of its simplicity and accuracy. The investigation of vibration problems of plates on elastic foundation have attracted the attention of many researchers working on structural foundation analysis and design. Lam *et al*. (2000) applied the Green's functions to achieve canonical exact solutions of elastic bending, buckling and vibration for Levy plates resting on two-parameter elastic foundations. The free and forced vibration analysis for a Reissner-Mindlin plate with four free edges resting on Pasternak type elastic foundation is studied by Shen *et al*. (2001). By employing the Rayleigh-Ritz method, the three dimensional vibration of rectangular thick plates on elastic foundations was investigated by Zhou *et al*. (2004). Zhong and Yin (2008) investigated the free vibration behavior of plate on Winkler foundation by finite integral transform method. Ferreira *et al*. (2010) used the radial basis function collocation method to study static deformation and free vibration of plates on Pasternak foundation. Free vibration analysis of moderately thick trapezoidal symmetrically laminated plates with various combinations of boundary conditions is studied by Zamani *et al*. (2012). Kumar and Lal (2012) studied the vibration analysis of nonhomogeneous orthotropic rectangular plates with bilinear thickness variation resting on Winkler foundation. A simple refined theory for bending, buckling, and vibration of thick plates resting on elastic foundation is introduced by Thai *et al*. (2013). Analytical solution of a refined plate theory is developed for free vibration analysis of functionally graded plates under various boundary conditions by Thai and Choi (2014). Nguyen-Thoi *et al*. (2014) presented an edge-based smoothed three-node Mindlin plate element (ES-MIN3) for static and free vibration analyses of plates. An original first shear deformation theory to study advanced composites on elastic foundation is presented by Mantari and Granados (2016). Uğurlu (2016) analyzed the vibration of elastic bottom plates of fluid storage tanks resting on Pasternak foundation based on boundary element method. A simplified first-order shear deformation theory for bending, buckling and free vibration analyses of isotropic plates on elastic foundations is investigated by Park and Choi (2017).

Furthermore, the mechanical behavior of concrete structures containing nanoparticles has been investigated experimentally by a number of researchers, but there is little mathematical works in this field. Investigations on the development of powder concrete with nano-SiO_2 particles are made by Jo *et al*. (2007). Fathi *et al*. (2017) investigated the mechanical and physical properties of expanded polystyrene structural concrete containing Micro-silica and Nano-silica. Effect of nanosilica on the compressive strength development and water absorption properties of cement paste and concrete containing Fly Ash is tested by Ehsani *et al*. (2017). In the field of mathematical modeling of concrete structures, Jafarian Arani and Kolahchi (2016) considered buckling analysis of concrete columns reinforced with carbon nanotubes by using Euler-Bernoulli and Timoshenko beam models. The nonlinear buckling of a concrete column reinforced with SiO2 nanoparticles is investigated by Zamanian *et al*. (2017). Also, Arbabi *et al*. (2017) studied the buckling of concrete columns reinforced with Zinc Oxide nanoparticles subjected to electric field. Mechanical characteristics of a classical concrete lightened by the addition of treated straws were presented by Kammoun and Trabelsi (2018). Effect of porosity on frost resistance of Portland cement pervious concrete was presented by Zhang *et al*. (2018).

It is worthy of noting that, this paper is mainly concerned with the vibration of concrete foundations reinforced by SiO_2 nanoparticles resting on soli medium. To the best of the authors' knowledge, the effects of using nano particles on the vibration of concrete foundations have not been investigated. In order to obtain the equivalent material properties of nano-composite structure, the Mori-Tanaka model is used. Also, higher-order shear deformation theories (HSDTs)

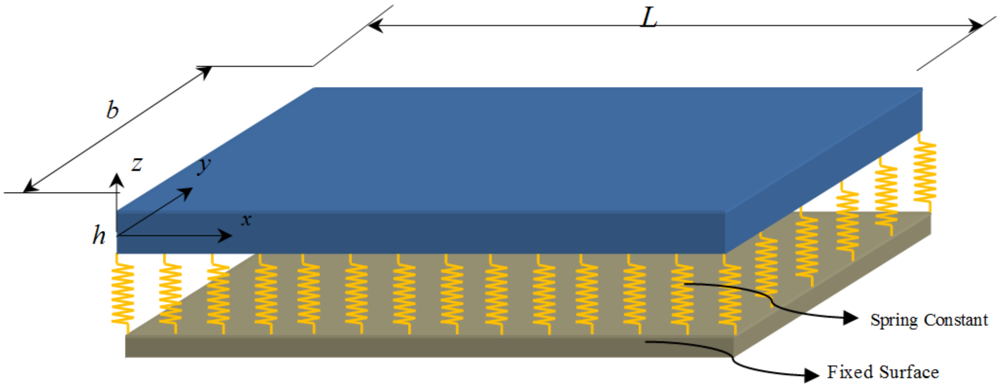

Fig. 1 A schematic figure for concrete foundation reinforced with SiO_2 nanoparticles resting on soil medium

were proposed to avoid the use of shear correction factor and obtain better prediction of response of thick foundation. Applying Reddy higher-order shear deformation theory, the motion equations are obtained based on Hamilton's principal. Also, Navier method is applied for obtaining the frequency of the system. The effects of volume percent and agglomeration of SiO_2 nanoparticles, soil medium and geometrical parameters of structure on the frequency of system are disused in detail.

2. Mathematical modeling

A concrete foundation reinforced with SiO_2 nanoparticles with length L, width b, concrete thickness h and is considered as shown in Fig. 1.

2.1 Reddy theory

Based on Reddy theory, the displacement field can be expressed as (Reddy 2002)

$$\begin{cases} u_1(x,y,z,t) = u(x,y,t) + z\,\phi_x(x,y,t) + c_1 z^3\left(\phi_x + \dfrac{\partial w}{\partial x}\right), \\ u_2(x,y,z,t) = v(x,y,t) + z\,\phi_y(x,y,t) + c_1 z^3\left(\phi_y + \dfrac{\partial w}{\partial y}\right), \\ u_3(x,y,z,t) = w(x,y,t), \end{cases} \tag{1}$$

where $(u_1(x,y,z,t), u_2(x,y,z,t), u_3(x,y,z,t))$ denote the displacement components at an arbitrary point (x,y,z) in the plate, and $(u(x,y,t), v(x,y,t), w(x,y,t))$ are the displacement of a material point at (x,y) on the mid-plane (i.e., $z=0$) of the plate along the x and y directions, respectively; ϕ_x and ϕ_y are the rotations of the normal to the mid-plane about x and y directions, respectively; Also, c_1=4/3h^2.

Based on above relations, the strain-displacement equations may be written as

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{pmatrix} + z \begin{pmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{pmatrix} + z^3 \begin{pmatrix} \varepsilon_{xx}^3 \\ \varepsilon_{yy}^3 \\ \gamma_{xy}^3 \end{pmatrix}, \tag{2a}$$

$$\begin{pmatrix} \gamma_{yz} \\ \gamma_{xz} \end{pmatrix} = \begin{pmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{pmatrix} + z^2 \begin{pmatrix} \gamma_{yz}^2 \\ \gamma_{xz}^2 \end{pmatrix}, \tag{2b}$$

where

$$\begin{pmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix}, \quad \begin{pmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{pmatrix} = \begin{pmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{pmatrix}, \quad \begin{pmatrix} \varepsilon_{xx}^3 \\ \varepsilon_{yy}^3 \\ \gamma_{xy}^3 \end{pmatrix} = c_1 \begin{pmatrix} \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y} \end{pmatrix}, \tag{2c}$$

$$\begin{pmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{pmatrix} = \begin{pmatrix} \phi_y + \frac{\partial w}{\partial y} \\ \phi_x + \frac{\partial w}{\partial x} \end{pmatrix}, \quad \begin{pmatrix} \gamma_{yz}^2 \\ \gamma_{xz}^2 \end{pmatrix} = c_2 \begin{pmatrix} \phi_y + \frac{\partial w}{\partial y} \\ \phi_x + \frac{\partial w}{\partial x} \end{pmatrix}, \tag{2d}$$

where $c_2=3c_1$.

2.2 Stress-strain relations

Based on Hook's law, we have

$$\sigma_{xx}^c = Q_{11}\varepsilon_{xx} + Q_{12}\varepsilon_{yy}, \tag{3}$$

$$\sigma_{yy}^c = Q_{12}\varepsilon_{xx} + Q_{22}\varepsilon_{yy}, \tag{4}$$

$$\tau_{yz}^c = Q_{44}\gamma_{yz}, \tag{5}$$

$$\tau_{xz}^c = Q_{55}\gamma_{zx}, \tag{6}$$

$$\tau_{xy}^c = Q_{66}\gamma_{xy}, \tag{7}$$

where Q_{ij} are elastic constants which can be obtained by Mori-Tanaka model.

2.3 Mori-Tanaka model and agglomeration effects

In this section, the effective modulus of the concrete foundation reinforced by SiO_2

nanoparticles is developed. Different methods are available to obtain the average properties of a composite (Mori and Tanaka 1973). Due to its simplicity and accuracy even at high volume fractions of the inclusions, the Mori-Tanaka method is employed in this section. The matrix is assumed to be isotropic and elastic, with the Young's modulus E_m and the Poisson's ratio υ_m. The constitutive relations for a layer of the composite with the principal axes parallel to the x, y and z directions are (Mori and Tanaka 1973)

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} k+m & l & k-m & 0 & 0 & 0 \\ l & n & l & 0 & 0 & 0 \\ k-m & l & k+m & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & p \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \tag{8}$$

where $\sigma_{ij}, \varepsilon_{ij}, \gamma_{ij}, k, m, n, l, p$ are the stress components, the strain components and the stiffness coefficients respectively. According to the Mori-Tanaka method the stiffness coefficients are given by Mori and Tanaka (1973)

$$\begin{aligned} k &= \frac{E_m\{E_m c_m + 2k_r(1+\nu_m)[1+c_r(1-2\nu_m)]\}}{2(1+\nu_m)[E_m(1+c_r-2\nu_m)+2c_m k_r(1-\nu_m-2\nu_m^2)]} \\ l &= \frac{E_m\{c_m\nu_m[E_m+2k_r(1+\nu_m)]+2c_r l_r(1-\nu_m^2)]\}}{(1+\nu_m)[E_m(1+c_r-2\nu_m)+2c_m k_r(1-\nu_m-2\nu_m^2)]} \\ n &= \frac{E_m^2 c_m(1+c_r-c_m\nu_m)+2c_m c_r(k_r n_r-l_r^2)(1+\nu_m)^2(1-2\nu_m)}{(1+\nu_m)[E_m(1+c_r-2\nu_m)+2c_m k_r(1-\nu_m-2\nu_m^2)]} \\ &\quad + \frac{E_m[2c_m^2 k_r(1-\nu_m)+c_r n_r(1+c_r-2\nu_m)-4c_m l_r\nu_m]}{E_m(1+c_r-2\nu_m)+2c_m k_r(1-\nu_m-2\nu_m^2)} \\ p &= \frac{E_m[E_m c_m+2p_r(1+\nu_m)(1+c_r)]}{2(1+\nu_m)[E_m(1+c_r)+2c_m p_r(1+\nu_m)]} \\ m &= \frac{E_m[E_m c_m+2m_r(1+\nu_m)(3+c_r-4\nu_m)]}{2(1+\nu_m)\{E_m[c_m+4c_r(1-\nu_m)]+2c_m m_r(3-\nu_m-4\nu_m^2)\}} \end{aligned} \tag{9}$$

where the subscripts m and r stand for matrix and reinforcement respectively. C_m and C_r are the volume fractions of the matrix and the nanoparticles respectively and k_r, l_r, n_r, p_r, m_r are the Hills elastic modulus for the nanoparticles (Mori and Tanaka, 1973). The experimental results show that the assumption of uniform dispersion for nanoparticles in the matrix is not correct and the most of nanoparticles are bent and centralized in one area of the matrix. These regions with concentrated nanoparticles are assumed to have spherical shapes, and are considered as ‘‘inclusions’’ with different elastic properties from the surrounding material. The total volume V_r of nanoparticles can be divided into the following two parts (Shi *et al*. 2004)

$$V_r = V_r^{inclusion} + V_r^m \tag{10}$$

where $V_r^{inclusion}$ and V_r^m are the volumes of nanoparticles dispersed in the spherical inclusions and in the matrix, respectively. Introduce two parameters ξ and ζ describe the agglomeration of nanoparticles

$$\xi = \frac{V_{inclusion}}{V}, \tag{11}$$

$$\zeta = \frac{V_r^{inclusion}}{V_r}. \tag{12}$$

However, the average volume fraction c_r of nanoparticles in the composite is

$$C_r = \frac{V_r}{V}. \tag{13}$$

Assume that all the orientations of the nanoparticles are completely random. Hence, the effective bulk modulus (K) and effective shear modulus (G) may be written as

$$K = K_{out}\left[1+\frac{\xi\left(\frac{K_{in}}{K_{out}}-1\right)}{1+\alpha(1-\xi)\left(\frac{K_{in}}{K_{out}}-1\right)}\right], \tag{14}$$

$$G = G_{out}\left[1+\frac{\xi\left(\frac{G_{in}}{G_{out}}-1\right)}{1+\beta(1-\xi)\left(\frac{G_{in}}{G_{out}}-1\right)}\right], \tag{15}$$

where

$$K_{in} = K_m + \frac{(\delta_r - 3K_m\chi_r)C_r\zeta}{3(\xi - C_r\zeta + C_r\zeta\chi_r)}, \tag{16}$$

$$K_{out} = K_m + \frac{C_r(\delta_r - 3K_m\chi_r)(1-\zeta)}{3\left[1-\xi - C_r(1-\zeta) + C_r\chi_r(1-\zeta)\right]}, \tag{17}$$

$$G_{in} = G_m + \frac{(\eta_r - 3G_m\beta_r)C_r\zeta}{2(\xi - C_r\zeta + C_r\zeta\beta_r)}, \tag{18}$$

$$G_{out} = G_m + \frac{C_r(\eta_r - 3G_m\beta_r)(1-\zeta)}{2\left[1-\xi - C_r(1-\zeta) + C_r\beta_r(1-\zeta)\right]}, \tag{19}$$

where χ_r, β_r, δ_r, η_r may be calculated as

$$\chi_r = \frac{3\left(K_m + G_m\right) + k_r - l_r}{3\left(k_r + G_m\right)}, \tag{20}$$

$$\beta_r = \frac{1}{5}\left\{\frac{4G_m + 2k_r + l_r}{3\left(k_r + G_m\right)} + \frac{4G_m}{\left(p_r + G_m\right)} + \frac{2\left[G_m\left(3K_m + G_m\right) + G_m\left(3K_m + 7G_m\right)\right]}{G_m\left(3K_m + G_m\right) + m_r\left(3K_m + 7G_m\right)}\right\}, \tag{21}$$

$$\delta_r = \frac{1}{3}\left[n_r + 2l_r + \frac{\left(2k_r - l_r\right)\left(3K_m + 2G_m - l_r\right)}{k_r + G_m}\right], \tag{22}$$

$$\eta_r = \frac{1}{5}\left[\frac{2}{3}\left(n_r - l_r\right) + \frac{4G_m p_r}{\left(p_r + G_m\right)} + \frac{8G_m m_r\left(3K_m + 4G_m\right)}{3K_m\left(m_r + G_m\right) + G_m\left(7m_r + G_m\right)} + \frac{2\left(k_r - l_r\right)\left(2G_m + l_r\right)}{3\left(k_r + G_m\right)}\right]. \tag{23}$$

where, K_m and G_m are the bulk and shear moduli of the matrix which can be written as

$$K_m = \frac{E_m}{3\left(1 - 2\upsilon_m\right)}, \tag{24}$$

$$G_m = \frac{E_m}{2\left(1 + \upsilon_m\right)}. \tag{25}$$

Furthermore, β, α can be obtained from

$$\alpha = \frac{\left(1 + \upsilon_{out}\right)}{3\left(1 - \upsilon_{out}\right)}, \tag{26a}$$

$$\beta = \frac{2\left(4 - 5\upsilon_{out}\right)}{15\left(1 - \upsilon_{out}\right)}, \tag{26b}$$

$$\upsilon_{out} = \frac{3K_{out} - 2G_{out}}{6K_{out} + 2G_{out}}. \tag{27}$$

Finally, the elastic modulus (E) and poison's ratio (υ) can be calculated as

$$E = \frac{9KG}{3K + G}, \tag{28}$$

$$\upsilon = \frac{3K - 2G}{6K + 2G}. \tag{29}$$

2.4 Energy method

The potential energy can be written as

$$U = \frac{1}{2}\int\left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{xy}\gamma_{xy} + \sigma_{xz}\gamma_{xz} + \sigma_{yz}\gamma_{yz}\right)dV \tag{30}$$

Combining of Eqs. (1), (3)-(7) and (30) yields

$$U = \frac{1}{2}\int \left(N_{xx}\left(\frac{\partial u}{\partial x}\right) + N_{yy}\left(\frac{\partial v}{\partial y}\right) + Q_{yy}\left(\frac{\partial w}{\partial y} + \phi_y\right) + Q_{xx}\left(\frac{\partial w}{\partial x} + \phi_x\right) + N_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \right.$$
$$+ M_{xx}\frac{\partial \phi_x}{\partial x} + M_{yy}\frac{\partial \phi_y}{\partial y} + M_{xy}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + K_{yy}\left(c_2\left(\phi_y + \frac{\partial w}{\partial y}\right)\right) + K_{xx}\left(c_2\left(\phi_x + \frac{\partial w}{\partial x}\right)\right) \tag{31}$$
$$\left. + P_{xx}\left(c_1\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)\right) + P_{yy}\left(c_1\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right)\right) + P_{xy}\left(\frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y} + 2\frac{\partial^2 w}{\partial x \partial y}\right)\right) dA,$$

where the stress resultant-displacement relations can be written as

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} dz, \tag{32}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} zdz, \tag{33}$$

$$\begin{Bmatrix} P_{xx} \\ P_{yy} \\ P_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} z^3 dz, \tag{34}$$

$$\begin{bmatrix} Q_{xx} \\ Q_{yy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} dz, \tag{35}$$

$$\begin{bmatrix} K_{xx} \\ K_{yy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} z^2 dz, \tag{36}$$

Substituting Eqs. (1) and (3)-(7) into Eqs. (32)-(36), the stress resultant-displacement relations can be obtained as follow

$$N_{xx} = A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} + A_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + B_{11}\frac{\partial \phi_x}{\partial x} + B_{12}\frac{\partial \phi_y}{\partial y} + B_{16}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$
$$+ E_{11}c_1\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + E_{12}c_1\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + E_{16}c_1\left(\frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y} + 2\frac{\partial^2 w}{\partial x \partial y}\right),$$
$$N_{yy} = A_{12}\frac{\partial u}{\partial x} + A_{22}\frac{\partial v}{\partial y} + A_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + B_{12}\frac{\partial \phi_x}{\partial x} + B_{22}\frac{\partial \phi_y}{\partial y} + B_{26}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$+E_{12}c_1\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+E_{22}c_1\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+E_{26}c_1\left(\frac{\partial\phi_y}{\partial x}+\frac{\partial\phi_x}{\partial y}+2\frac{\partial^2 w}{\partial x\partial y}\right),$$

$$N_{xy}=A_{16}\frac{\partial u}{\partial x}+A_{26}\frac{\partial v}{\partial y}+A_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+B_{16}\frac{\partial\phi_x}{\partial x}+B_{26}\frac{\partial\phi_y}{\partial y}+B_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right) \tag{37}$$

$$+E_{16}c_1\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+E_{26}c_1\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+E_{66}c_1\left(\frac{\partial\phi_y}{\partial x}+\frac{\partial\phi_x}{\partial y}+2\frac{\partial^2 w}{\partial x\partial y}\right),$$

$$\begin{aligned}
M_{xx}&=B_{11}\frac{\partial u}{\partial x}+B_{12}\frac{\partial v}{\partial y}+B_{16}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}+D_{16}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\\
&+F_{11}c_1\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+F_{12}c_1\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+F_{16}c_1\left(\frac{\partial\phi_y}{\partial x}+\frac{\partial\phi_x}{\partial y}+2\frac{\partial^2 w}{\partial x\partial y}\right),\\
M_{yy}&=B_{12}\frac{\partial u}{\partial x}+B_{22}\frac{\partial v}{\partial y}+B_{26}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+D_{12}\frac{\partial\phi_x}{\partial x}+D_{22}\frac{\partial\phi_y}{\partial y}+D_{26}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\\
&+F_{12}c_1\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+F_{22}c_1\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+F_{26}c_1\left(\frac{\partial\phi_y}{\partial x}+\frac{\partial\phi_x}{\partial y}+2\frac{\partial^2 w}{\partial x\partial y}\right),\\
M_{xy}&=B_{16}\frac{\partial u}{\partial x}+B_{26}\frac{\partial v}{\partial y}+B_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+D_{16}\frac{\partial\phi_x}{\partial x}+D_{26}\frac{\partial\phi_y}{\partial y}+D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\\
&+F_{16}c_1\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+F_{26}c_1\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+F_{66}c_1\left(\frac{\partial\phi_y}{\partial x}+\frac{\partial\phi_x}{\partial y}+2\frac{\partial^2 w}{\partial x\partial y}\right),
\end{aligned} \tag{38}$$

$$\begin{aligned}
P_{xx}&=E_{11}\frac{\partial u}{\partial x}+E_{12}\frac{\partial v}{\partial y}+E_{16}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+F_{11}\frac{\partial\phi_x}{\partial x}+F_{12}\frac{\partial\phi_y}{\partial y}+F_{16}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\\
&+H_{11}c_1\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+H_{12}c_1\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+H_{16}c_1\left(\frac{\partial\phi_y}{\partial x}+\frac{\partial\phi_x}{\partial y}+2\frac{\partial^2 w}{\partial x\partial y}\right),\\
P_{yy}&=E_{12}\frac{\partial u}{\partial x}+E_{22}\frac{\partial v}{\partial y}+E_{26}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+F_{12}\frac{\partial\phi_x}{\partial x}+F_{22}\frac{\partial\phi_y}{\partial y}+F_{26}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\\
&+H_{12}c_1\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+H_{22}c_1\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+H_{26}c_1\left(\frac{\partial\phi_y}{\partial x}+\frac{\partial\phi_x}{\partial y}+2\frac{\partial^2 w}{\partial x\partial y}\right),\\
P_{xy}&=E_{16}\frac{\partial u}{\partial x}+E_{26}\frac{\partial v}{\partial y}+E_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+F_{16}\frac{\partial\phi_x}{\partial x}+F_{26}\frac{\partial\phi_y}{\partial y}+F_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\\
&+H_{16}c_1\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+H_{26}c_1\left(\frac{\partial\phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+H_{66}c_1\left(\frac{\partial\phi_y}{\partial x}+\frac{\partial\phi_x}{\partial y}+2\frac{\partial^2 w}{\partial x\partial y}\right),
\end{aligned} \tag{39}$$

$$Q_{xx}=A_{55}\left(\frac{\partial w}{\partial x}+\phi_x\right)+A_{45}\left(\frac{\partial w}{\partial y}+\phi_y\right)+D_{55}c_2\left(\phi_x+\frac{\partial w}{\partial x}\right)+D_{45}c_2\left(\frac{\partial w}{\partial y}+\phi_y\right),$$
$$Q_{yy}=A_{45}\left(\frac{\partial w}{\partial y}+\phi_y\right)+A_{44}\left(\frac{\partial w}{\partial y}+\phi_y\right)+D_{45}c_2\left(\phi_y+\frac{\partial w}{\partial y}\right)+D_{44}c_2\left(\frac{\partial w}{\partial y}+\phi_y\right), \tag{40}$$

$$K_{xx}=D_{55}\left(\frac{\partial w}{\partial x}+\phi_x\right)+D_{45}\left(\frac{\partial w}{\partial y}+\phi_y\right)+F_{55}c_2\left(\phi_x+\frac{\partial w}{\partial x}\right)+F_{45}c_2\left(\frac{\partial w}{\partial y}+\phi_y\right),$$
$$K_{yy}=D_{45}\left(\frac{\partial w}{\partial y}+\phi_y\right)+D_{44}\left(\frac{\partial w}{\partial y}+\phi_y\right)+F_{45}c_2\left(\phi_y+\frac{\partial w}{\partial y}\right)+F_{44}c_2\left(\frac{\partial w}{\partial y}+\phi_y\right), \tag{41}$$

where

$$A_{ij}=\int_{-h/2}^{h/2}Q_{ij}dz, \qquad (i,j=1,2,6) \tag{42}$$

$$B_{ij}=\int_{-h/2}^{h/2}Q_{ij}zdz, \tag{43}$$

$$D_{ij}=\int_{-h/2}^{h/2}Q_{ij}z^2dz, \tag{44}$$

$$E_{ij}=\int_{-h/2}^{h/2}Q_{ij}z^3dz, \tag{45}$$

$$F_{ij}=\int_{-h/2}^{h/2}Q_{ij}z^4dz, \tag{46}$$

$$H_{ij}=\int_{-h/2}^{h/2}Q_{ij}z^6dz, \tag{47}$$

The kinetic energy of system may be written as

$$U=\frac{\rho}{2}\int\left(\dot{u}_1^2+\dot{u}_2^2+\dot{u}_3^2\right)dV, \tag{48}$$

The external work due to Pasternak medium can be written as (Bowles 1988)

$$W_e=\int\int\left(-K_w w\right)wdA, \tag{49}$$

where K_w is Winkler's spring modulus. The governing equations can be derived by Hamilton's principal as follows

$$\int_0^t(\delta U-\delta K-\delta W_e)dt=0. \tag{50}$$

Substituting Eqs. (31), (48) and (49) into Eq. (50) yields the following governing equations

$$\delta u:\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}=I_0\frac{\partial^2 u}{\partial t^2}+J_1\frac{\partial^2\phi_x}{\partial t^2}-\frac{4I_3}{h^2}\frac{\partial^3 w}{\partial t^2\partial x}, \tag{51}$$

$$\delta v:\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_{yy}}{\partial y}=I_0\frac{\partial^2 v}{\partial t^2}+J_1\frac{\partial^2\phi_y}{\partial t^2}-\frac{4I_3}{h^2}\frac{\partial^3 w}{\partial t^2\partial y}, \tag{52}$$

$$\begin{aligned}&\delta w:\frac{\partial Q_{xx}}{\partial x}+\frac{\partial Q_{yy}}{\partial y}+c_2\left(\frac{\partial K_{xx}}{\partial x}+\frac{\partial K_{yy}}{\partial y}\right)+N_{xx}\frac{\partial^2 w}{\partial x^2}+N_{yy}\frac{\partial^2 w}{\partial y^2}\\&-c_1\left(\frac{\partial^2 P_{xx}}{\partial x^2}+2\frac{\partial^2 P_{xy}}{\partial x\partial y}+\frac{\partial^2 P_{yy}}{\partial y^2}\right)-K_w w=I_0\frac{\partial^2 w}{\partial t^2}-\left(\frac{4}{3h^2}\right)^2 I_6\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^4 w}{\partial y^2\partial t^2}\right)\\&+\frac{4}{3h^2}\left(I_3\frac{\partial^3 u}{\partial t^2\partial x}+I_3\frac{\partial^3 v}{\partial t^2\partial y}+J_4\left(\frac{\partial^3\phi_x}{\partial t^2\partial x}+\frac{\partial^3\phi_y}{\partial t^2\partial y}\right)\right),\end{aligned} \tag{53}$$

$$\begin{aligned}&\delta\phi_x:\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}+c_1\left(\frac{\partial P_{xx}}{\partial x}+\frac{\partial P_{xy}}{\partial y}\right)-Q_{xx}-c_2K_{xx}\\&=J_1\frac{\partial^2 u}{\partial t^2}+K_2\frac{\partial^2\phi_x}{\partial t^2}-\frac{4}{3h^2}J_4\frac{\partial^3 w}{\partial t^2\partial x},\end{aligned} \tag{54}$$

$$\begin{aligned}&\delta\phi_y:\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{yy}}{\partial y}+c_1\left(\frac{\partial P_{xy}}{\partial x}+\frac{\partial P_{yy}}{\partial y}\right)-Q_{yy}-c_2K_{yy}\\&=J_1\frac{\partial^2 v}{\partial t^2}+K_2\frac{\partial^2\phi_y}{\partial t^2}-\frac{4}{3h^2}J_4\frac{\partial^3 w}{\partial t^2\partial y},\end{aligned} \tag{55}$$

where

$$I_i=\int_{-h/2}^{h/2}\rho z^i dz \qquad (i=0,1,\ldots,6), \tag{56}$$

$$J_i=I_i-\frac{4}{3h^2}I_{i+2} \qquad (i=1,4), \tag{57}$$

$$K_2=I_2-\frac{8}{3h^2}I_4+\left(\frac{4}{3h^2}\right)^2 I_6, \tag{58}$$

Substituting Eqs. (37) to (41) into Eqs. (51) to (55), the governing equations can be written as follow

$$\begin{aligned}&A_{11}\frac{\partial^2 u}{\partial x^2}+A_{12}\frac{\partial^2 v}{\partial x\partial y}+A_{16}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right)+B_{11}\frac{\partial^2\varphi_x}{\partial x^2}+B_{12}\frac{\partial^2\varphi_y}{\partial x\partial y}\\&+B_{16}\left(\frac{\partial^2\varphi_x}{\partial x\partial y}+\frac{\partial^2\varphi_y}{\partial x^2}\right)+E_{11}c_1\left(\frac{\partial^2\varphi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)+E_{12}c_1\left(\frac{\partial^2\varphi_y}{\partial x\partial y}+\frac{\partial^3 w}{\partial x\partial y^2}\right)\\&+E_{16}c_1\left(\frac{\partial^2\varphi_y}{\partial x^2}+\frac{\partial^2\varphi_x}{\partial x\partial y}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)+A_{16}\frac{\partial^2 u}{\partial x\partial y}+A_{26}\frac{\partial^2 v}{\partial y^2}+A_{66}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)\\&+B_{16}\frac{\partial^2\varphi_x}{\partial x\partial y}+B_{26}\frac{\partial^2\varphi_y}{\partial y^2}+B_{66}\left(\frac{\partial^2\varphi_x}{\partial y^2}+\frac{\partial^2\varphi_y}{\partial x\partial y}\right)+E_{16}c_1\left(\frac{\partial^2\varphi_x}{\partial x\partial y}+\frac{\partial^3 w}{\partial y\partial x^2}\right)\end{aligned}$$

$$+E_{26}c_1\left(\frac{\partial^2\varphi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)+E_{66}c_1\left(\frac{\partial^2\varphi_y}{\partial x\,\partial y}+\frac{\partial^2\varphi_x}{\partial y^2}+2\frac{\partial^3 w}{\partial x\,\partial y^2}\right)=I_0\frac{\partial^2 u}{\partial t^2}+J_1\frac{\partial^2\phi_x}{\partial t^2}-\frac{4I_3}{h^2}\frac{\partial^3 w}{\partial t^2\partial x},\tag{59}$$

$$\begin{aligned}
&A_{16}\frac{\partial^2 u}{\partial x^2}+A_{26}\frac{\partial^2 v}{\partial x\,\partial y}+A_{66}\left(\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial x^2}\right)+B_{16}\frac{\partial^2\varphi_x}{\partial x^2}+B_{26}\frac{\partial^2\varphi_y}{\partial x\,\partial y}\\
&+B_{66}\left(\frac{\partial^2\varphi_x}{\partial x\,\partial y}+\frac{\partial^2\varphi_y}{\partial x^2}\right)+E_{16}c_1\left(\frac{\partial^2\varphi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)+E_{26}c_1\left(\frac{\partial^2\varphi_y}{\partial x\,\partial y}+\frac{\partial^3 w}{\partial x\,\partial y^2}\right)\\
&+E_{66}c_1\left(\frac{\partial^2\varphi_y}{\partial x^2}+\frac{\partial^2\varphi_x}{\partial x\,\partial y}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)+A_{21}\frac{\partial^2 u}{\partial x\,\partial y}+A_{22}\frac{\partial^2 v}{\partial y^2}+A_{26}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\,\partial y}\right)\\
&+B_{21}\frac{\partial^2\varphi_x}{\partial x\,\partial y}+B_{22}\frac{\partial^2\varphi_y}{\partial y^2}+B_{26}\left(\frac{\partial^2\varphi_x}{\partial y^2}+\frac{\partial^2\varphi_y}{\partial x\,\partial y}\right)+E_{21}c_1\left(\frac{\partial^2\varphi_x}{\partial x\,\partial y}+\frac{\partial^3 w}{\partial y\,\partial x^2}\right)\\
&+E_{22}c_1\left(\frac{\partial^2\varphi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)+E_{26}c_1\left(\frac{\partial^2\varphi_y}{\partial x\,\partial y}+\frac{\partial^2\varphi_x}{\partial y^2}+2\frac{\partial^3 w}{\partial x\,\partial y^2}\right)=I_0\frac{\partial^2 v}{\partial t^2}+J_1\frac{\partial^2\phi_y}{\partial t^2}-\frac{4I_3}{h^2}\frac{\partial^3 w}{\partial t^2\partial y},
\end{aligned}\tag{60}$$

$$\begin{aligned}
&A_{55}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\varphi_x}{\partial x}\right)+A_{45}\left(\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\partial\varphi_y}{\partial x}\right)+D_{55}c_2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\varphi_x}{\partial x}\right)+D_{45}c_2\left(\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\partial\varphi_y}{\partial x}\right)\\
&+A_{45}\left(\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\partial\varphi_x}{\partial y}\right)+A_{44}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\varphi_y}{\partial y}\right)+D_{45}c_2\left(\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\partial\varphi_x}{\partial y}\right)+D_{44}c_2\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\varphi_y}{\partial y}\right)\\
&+c_2\left(\begin{aligned}&D_{55}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\varphi_x}{\partial x}\right)+D_{45}\left(\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\partial\varphi_y}{\partial x}\right)+F_{55}c_2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\varphi_x}{\partial x}\right)+F_{45}c_2\left(\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\partial\varphi_y}{\partial x}\right)\\&+D_{45}\left(\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\partial\varphi_x}{\partial y}\right)+D_{44}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\varphi_y}{\partial y}\right)+F_{45}c_2\left(\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\partial\varphi_x}{\partial y}\right)+F_{44}c_2\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\varphi_y}{\partial y}\right)\end{aligned}\right)\\
&+N_{xx}\frac{\partial^2 w}{\partial x^2}+N_{yy}\frac{\partial^2 w}{\partial y^2}-c_1\left(E_{11}\frac{\partial^3 u}{\partial x^3}+E_{12}\frac{\partial^3 v}{\partial x^2\partial y}+E_{16}\left(\frac{\partial^3 u}{\partial x^2\partial y}+\frac{\partial^3 v}{\partial x^3}\right)+F_{11}\frac{\partial^3\varphi_x}{\partial x^3}\right.\\
&+F_{12}\frac{\partial^3\varphi_y}{\partial x^2\partial y}+F_{16}\left(\frac{\partial^3\varphi_x}{\partial x^2\partial y}+\frac{\partial^3\varphi_y}{\partial x^3}\right)+H_{11}c_1\left(\frac{\partial^3\varphi_x}{\partial x^3}+\frac{\partial^4 w}{\partial x^4}\right)+H_{12}c_1\left(\frac{\partial^3\varphi_y}{\partial x^2\partial y}+\frac{\partial^4 w}{\partial x^2\partial y^2}\right)\\
&+H_{16}c_1\left(\frac{\partial^3\varphi_y}{\partial x^3}+\frac{\partial^3\varphi_x}{\partial x^2\partial y}+2\frac{\partial^4 w}{\partial x^3\partial y}\right)+E_{12}\frac{\partial^3 u}{\partial y^2\partial x}+E_{22}\frac{\partial^3 v}{\partial y^3}+E_{26}\left(\frac{\partial^3 u}{\partial y^3}+\frac{\partial^3 v}{\partial x\,\partial y^2}\right)\\
&+F_{12}\frac{\partial^3\varphi_x}{\partial x\,\partial y^2}+F_{22}\frac{\partial^3\varphi_y}{\partial y^3}+F_{26}\left(\frac{\partial^3\varphi_x}{\partial y^3}+\frac{\partial^3\varphi_y}{\partial x\,\partial y^2}\right)+H_{12}c_1\left(\frac{\partial^3\varphi_x}{\partial x\,\partial y^2}+\frac{\partial^4 w}{\partial x^2\partial y^2}\right)\\
&+H_{22}c_1\left(\frac{\partial^3\varphi_y}{\partial y^3}+\frac{\partial^4 w}{\partial y^4}\right)+H_{26}c_1\left(\frac{\partial^3\varphi_y}{\partial x\,\partial y^2}+\frac{\partial^3\varphi_x}{\partial y^3}+2\frac{\partial^4 w}{\partial x\,\partial y^3}\right)+2E_{16}\frac{\partial^3 u}{\partial y\,\partial x^2}+2E_{26}\frac{\partial^3 v}{\partial y^2\partial x}\\
&+2E_{66}\left(\frac{\partial^3 u}{\partial y^2\partial x}+\frac{\partial^3 v}{\partial x^2\partial y}\right)+2F_{16}\frac{\partial^3\varphi_x}{\partial x^2\partial y}+2F_{26}\frac{\partial^3\varphi_y}{\partial y^2\partial x}+2F_{66}\left(\frac{\partial^3\varphi_x}{\partial y^2\partial x}+\frac{\partial^3\varphi_y}{\partial x^2\partial y}\right)\\
&+2H_{16}c_1\left(\frac{\partial^3\varphi_x}{\partial x^2\partial y}+\frac{\partial^4 w}{\partial x^3\partial y}\right)+2H_{26}c_1\left(\frac{\partial^3\varphi_y}{\partial x\,\partial y^2}+\frac{\partial^4 w}{\partial y^3\partial x}\right)\\
&\left.+2H_{66}c_1\left(\frac{\partial^3\varphi_y}{\partial x^2\partial y}+\frac{\partial^3\varphi_x}{\partial y^2\partial x}+2\frac{\partial^4 w}{\partial x^2\partial y^2}\right)\right)-k_w w=I_0\frac{\partial^2 w}{\partial t^2}-\left(\frac{4}{3h^2}\right)^2 I_6\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^4 w}{\partial y^2\partial t^2}\right)\\
&+\frac{4}{3h^2}\left(I_3\frac{\partial^3 u}{\partial t^2\partial x}+I_3\frac{\partial^3 v}{\partial t^2\partial y}+J_4\left(\frac{\partial^3\phi_x}{\partial t^2\partial x}+\frac{\partial^3\phi_y}{\partial t^2\partial y}\right)\right),
\end{aligned}\tag{61}$$

$$
\begin{aligned}
& B_{11}\frac{\partial^2 u}{\partial x^2}+B_{12}\frac{\partial^2 v}{\partial x\,\partial y}+B_{16}\left(\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial x^2}\right)+D_{11}\frac{\partial^2 \varphi_x}{\partial x^2}+D_{12}\frac{\partial^2 \varphi_y}{\partial x\,\partial y}+D_{16}\left(\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+\frac{\partial^2 \varphi_y}{\partial x^2}\right)\\
& +F_{11}c_1\left(\frac{\partial^2 \varphi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)+F_{12}c_1\left(\frac{\partial^2 \varphi_y}{\partial x\,\partial y}+\frac{\partial^3 w}{\partial x\,\partial y^2}\right)+F_{16}c_1\left(\frac{\partial^2 \varphi_y}{\partial x^2}+\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)\\
& +B_{16}\frac{\partial^2 u}{\partial x\,\partial y}+B_{26}\frac{\partial^2 v}{\partial y^2}+B_{66}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\,\partial y}\right)+D_{16}\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+D_{26}\frac{\partial^2 \varphi_y}{\partial y^2}\\
& +D_{66}\left(\frac{\partial^2 \varphi_x}{\partial y^2}+\frac{\partial^2 \varphi_y}{\partial x\,\partial y}\right)+F_{16}c_1\left(\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+\frac{\partial^3 w}{\partial y\,\partial x^2}\right)+F_{26}c_1\left(\frac{\partial^2 \varphi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)\\
& +F_{66}c_1\left(\frac{\partial^2 \varphi_y}{\partial x\,\partial y}+\frac{\partial^2 \varphi_x}{\partial y^2}+2\frac{\partial^3 w}{\partial x\,\partial y^2}\right)+c_1\Bigg(E_{11}\frac{\partial^2 u}{\partial x^2}+E_{12}\frac{\partial^2 v}{\partial x\,\partial y}+E_{16}\left(\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial x^2}\right)\\
& +F_{11}\frac{\partial^2 \varphi_x}{\partial x^2}+F_{12}\frac{\partial^2 \varphi_y}{\partial x\,\partial y}+F_{16}\left(\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+\frac{\partial^2 \varphi_y}{\partial x^2}\right)+H_{11}c_1\left(\frac{\partial^2 \varphi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)\\
& +H_{12}c_1\left(\frac{\partial^2 \varphi_y}{\partial x\,\partial y}+\frac{\partial^3 w}{\partial x\,\partial y^2}\right)+H_{16}c_1\left(\frac{\partial^2 \varphi_y}{\partial x^2}+\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)+E_{16}\frac{\partial^2 u}{\partial x\,\partial y}+E_{26}\frac{\partial^2 v}{\partial y^2}\\
& +E_{66}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\,\partial y}\right)+F_{16}\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+F_{26}\frac{\partial^2 \varphi_y}{\partial y^2}+F_{66}\left(\frac{\partial^2 \varphi_x}{\partial y^2}+\frac{\partial^2 \varphi_y}{\partial x\,\partial y}\right)\\
& +H_{16}c_1\left(\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+\frac{\partial^3 w}{\partial y\,\partial x^2}\right)+H_{26}c_1\left(\frac{\partial^2 \varphi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)+H_{66}c_1\left(\frac{\partial^2 \varphi_y}{\partial x\,\partial y}+\frac{\partial^2 \varphi_x}{\partial y^2}+2\frac{\partial^3 w}{\partial x\,\partial y^2}\right)\\
& -A_{55}\left(\frac{\partial w}{\partial x}+\varphi_x\right)-A_{45}\left(\frac{\partial w}{\partial y}+\varphi_y\right)-D_{55}c_2\left(\varphi_x+\frac{\partial w}{\partial x}\right)-D_{45}c_2\left(\frac{\partial w}{\partial y}+\varphi_y\right)\\
& +c_2\left(-D_{55}\left(\frac{\partial w}{\partial x}+\varphi_x\right)-D_{45}\left(\frac{\partial w}{\partial y}+\varphi_y\right)-F_{55}c_2\left(\varphi_x+\frac{\partial w}{\partial x}\right)-F_{45}c_2\left(\frac{\partial w}{\partial y}+\varphi_y\right)\right)\Bigg)\\
& =J_1\frac{\partial^2 u}{\partial t^2}+K_2\frac{\partial^2 \phi_x}{\partial t^2}-\frac{4}{3h^2}J_4\frac{\partial^3 w}{\partial t^2\partial x},
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
& B_{16}\frac{\partial^2 u}{\partial x^2}+B_{26}\frac{\partial^2 v}{\partial x\,\partial y}+B_{66}\left(\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial x^2}\right)+D_{16}\frac{\partial^2 \varphi_x}{\partial x^2}+D_{26}\frac{\partial^2 \varphi_y}{\partial x\,\partial y}+D_{66}\left(\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+\frac{\partial^2 \varphi_y}{\partial x^2}\right)\\
& +F_{16}c_1\left(\frac{\partial^2 \varphi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)+F_{26}c_1\left(\frac{\partial^2 \varphi_y}{\partial x\,\partial y}+\frac{\partial^3 w}{\partial x\,\partial y^2}\right)+F_{66}c_1\left(\frac{\partial^2 \varphi_y}{\partial x^2}+\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)\\
& +B_{12}\frac{\partial^2 u}{\partial x\,\partial y}+B_{22}\frac{\partial^2 v}{\partial y^2}+B_{26}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\,\partial y}\right)+D_{12}\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+D_{22}\frac{\partial^2 \varphi_y}{\partial y^2}\\
& +D_{26}\left(\frac{\partial^2 \varphi_x}{\partial y^2}+\frac{\partial^2 \varphi_y}{\partial x\,\partial y}\right)+F_{12}c_1\left(\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+\frac{\partial^3 w}{\partial y\,\partial x^2}\right)+F_{22}c_1\left(\frac{\partial^2 \varphi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)\\
& +F_{26}c_1\left(\frac{\partial^2 \varphi_y}{\partial x\,\partial y}+\frac{\partial^2 \varphi_x}{\partial y^2}+2\frac{\partial^3 w}{\partial x\,\partial y^2}\right)+c_1\Bigg(E_{16}\frac{\partial^2 u}{\partial x^2}+E_{26}\frac{\partial^2 v}{\partial x\,\partial y}+E_{66}\left(\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial x^2}\right)\\
& +F_{16}\frac{\partial^2 \varphi_x}{\partial x^2}+F_{26}\frac{\partial^2 \varphi_y}{\partial x\,\partial y}+F_{66}\left(\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+\frac{\partial^2 \varphi_y}{\partial x^2}\right)+H_{16}c_1\left(\frac{\partial^2 \varphi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)\\
& +H_{26}c_1\left(\frac{\partial^2 \varphi_y}{\partial x\,\partial y}+\frac{\partial^3 w}{\partial x\,\partial y^2}\right)+H_{66}c_1\left(\frac{\partial^2 \varphi_y}{\partial x^2}+\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)+E_{12}\frac{\partial^2 u}{\partial x\,\partial y}+E_{22}\frac{\partial^2 v}{\partial y^2}\\
& +E_{26}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\,\partial y}\right)+F_{12}\frac{\partial^2 \varphi_x}{\partial x\,\partial y}+F_{22}\frac{\partial^2 \varphi_y}{\partial y^2}+F_{26}\left(\frac{\partial^2 \varphi_x}{\partial y^2}+\frac{\partial^2 \varphi_y}{\partial x\,\partial y}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+H_{12}c_1\left(\frac{\partial^2\varphi_x}{\partial x\,\partial y}+\frac{\partial^3 w}{\partial y\,\partial x^2}\right)+H_{22}c_1\left(\frac{\partial^2\varphi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)+H_{26}c_1\left(\frac{\partial^2\varphi_y}{\partial x\,\partial y}+\frac{\partial^2\varphi_x}{\partial y^2}+2\frac{\partial^3 w}{\partial x\,\partial y^2}\right)\\
&-A_{45}\left(\frac{\partial w}{\partial x}+\varphi_x\right)-A_{44}\left(\frac{\partial w}{\partial y}+\varphi_y\right)-D_{45}c_2\left(\varphi_x+\frac{\partial w}{\partial x}\right)-D_{44}c_2\left(\frac{\partial w}{\partial y}+\varphi_y\right)\\
&+c_2\left(-D_{45}\left(\frac{\partial w}{\partial x}+\varphi_x\right)-D_{44}\left(\frac{\partial w}{\partial y}+\varphi_y\right)-F_{45}c_2\left(\varphi_x+\frac{\partial w}{\partial x}\right)-F_{44}c_2\left(\frac{\partial w}{\partial y}+\varphi_y\right)\right)\\
&=J_1\frac{\partial^2 v}{\partial t^2}+K_2\frac{\partial^2\phi_y}{\partial t^2}-\frac{4}{3h^2}J_4\frac{\partial^3 w}{\partial t^2\partial y},
\end{aligned}
\tag{63}
$$

3. Solution procedure

Steady state solutions to the governing equations of the system motion which relate to the simply supported boundary conditions can be assumed as

$$u(x,y,t)=u_0\cos(\frac{n\pi x}{L})\sin(\frac{m\pi y}{b})e^{i\omega t}, \tag{64}$$

$$v(x,y,t)=v_0\sin(\frac{n\pi x}{L})\cos(\frac{m\pi y}{b})e^{i\omega t}, \tag{65}$$

$$w(x,\mathrm{y},t)=w_0\sin(\frac{n\pi x}{L})\sin(\frac{m\pi y}{b})e^{i\omega t}, \tag{66}$$

$$\phi_x(x,y,t)=\psi_{x0}\cos(\frac{n\pi x}{L})\sin(\frac{m\pi y}{b})e^{i\omega t}, \tag{67}$$

$$\phi_y(x,y,t)=\psi_{y0}\sin(\frac{n\pi x}{L})\cos(\frac{m\pi y}{b})e^{i\omega t}, \tag{68}$$

Substituting Eqs. (64)-(68) into Eqs. (59)-(63) yields

$$
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15}\\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25}\\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35}\\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45}\\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55}
\end{bmatrix}
\begin{bmatrix}
u_0\\ v_0\\ w_0\\ \psi_{x0}\\ \psi_{y0}
\end{bmatrix}=0, \tag{69}
$$

where K_{ij} are defined in Appendix A. Finally, for calculating the frequency of the system (ω), the determinant of matrix in Eq. (69) should be equal to zero.

4. Numerical results and discussion

A computer program coded in Matlab is prepared for the vibration of concrete foundation

Table 1 Validation of present study with the other works

Method	Mode number			
	1	2	3	4
Whitney (1987)	15.171	33.248	44.387	60.682
Seçgin and Sarıgül (2008)	15.171	33.248	44.387	60.682
Dai *et al*. (2004)	15.17	33.32	44.51	60.78
Chen *et al*. (2003)	15.18	33.34	44.51	60.78
Chow *et al*. (1992)	15.19	33.31	44.52	60.79
Present	15.169	33.241	44.382	60.674

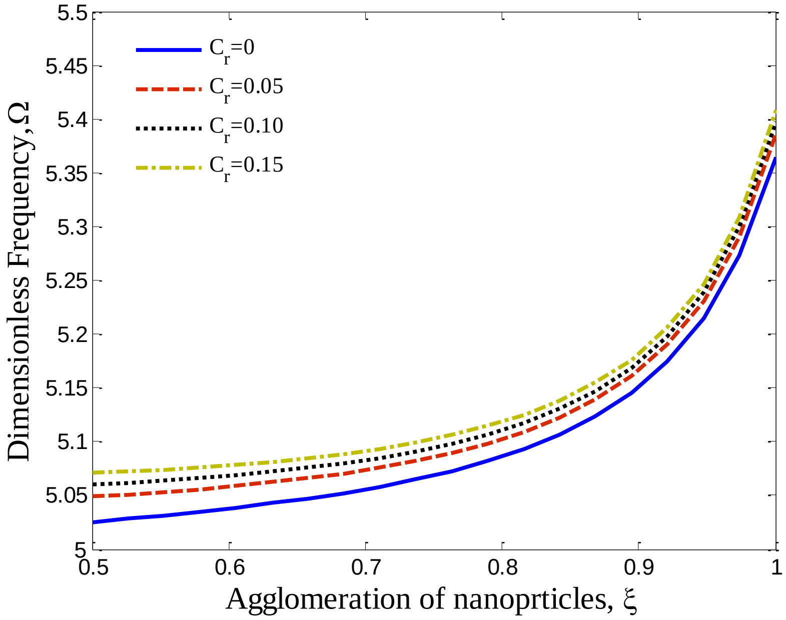

Fig. 2 Effects of SiO_2 nanoparticles volume percent on the dimensionless frequency versus agglomeration percent

reinforced with SiO_2 nanoparticles. SiO_2 nanoparticles have Yong's modulus of E_r=70 GPa and Poisson's ratio of v_r=0.2.

4.1 Validation

In this paper, to validate the results, the frequency of the structure is obtained by assuming the absence of soil medium (K_w=0). Therefore, all the mechanical properties and type of loading are the same as Whitney (1987). So the non-dimensional frequency is considered as $\Omega = \sqrt{\frac{\rho h \omega^2 L^4}{D_0}}$ in which $D_0 = E_1 h^3/(12(1-v_{12}v_{21}))$. The results are compared with five references which have used different solution method. Whitney (1987) is used exact solution while Seçgin and Sarıgül (2008) are applied discrete singular convolution approach. The numerical solution method of Dai *et al*. (2007), Chen *et al*. (2003), Chow *et al*. (1992) are mesh-free, finite element and Ritz, respectively.

Table 2 Spring constants of different soils

Soil	K_w (N/m^3)
Loose sand	4800-16000
Dense sand	64000-128000
Clayey medium dense sand	32000-80000
Clayey soil	12000-24000

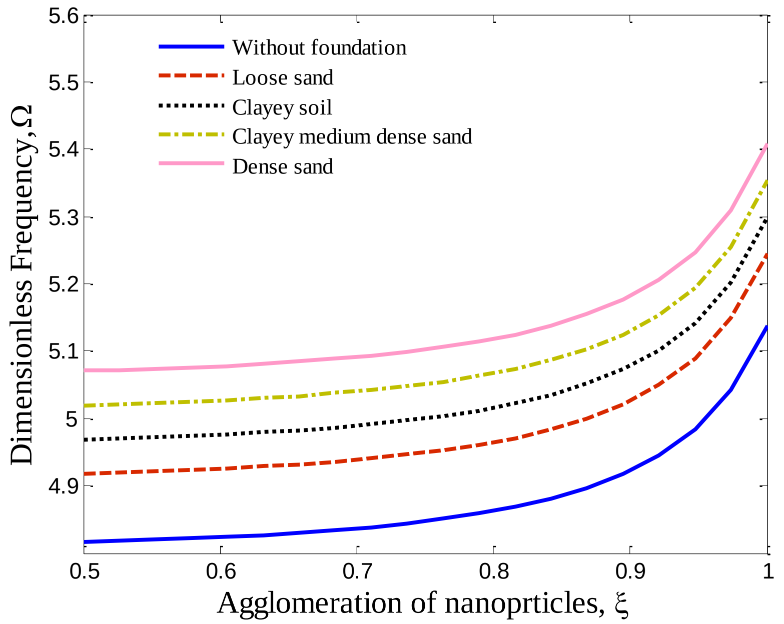

Fig. 3 Effects of soil medium on the dimensionless frequency versus agglomeration percent

As it is observed in Table 1 the results of present work are in accordance with the mentioned references.

4.2 Effects of different parameters

Fig. 2 illustrates the effect of the SiO_2 nanoparticles volume fraction on the dimensionless frequency of structure ($\Omega = \omega L\sqrt{\rho_m / E_m}$) versus agglomeration percent of SiO_2 nanoparticles. It can be seen that with increasing the values of SiO_2 nanoparticles volume fraction, the frequency of the system is increased. This is due to the fact that the increase of SiO_2 nanoparticles leads to a harder structure. In addition, decreasing the agglomeration of SiO_2 nanoparticles ($\xi \to 1$), the frequency is increased due to more stability of the foundation.

The dimensionless frequency of the nano-composite concrete foundation versus agglomeration percent of SiO_2 nanoparticles is demonstrated in Fig. 3 for different soil mediums. In this figure, four cases of loose sand, dense sand, Clayey medium dense sand and Clayey soil are considered with the spring constants of Table 2. As can be seen, considering soil medium increases the frequency of the structure. It is due to the fact that considering soil medium leads to stiffer

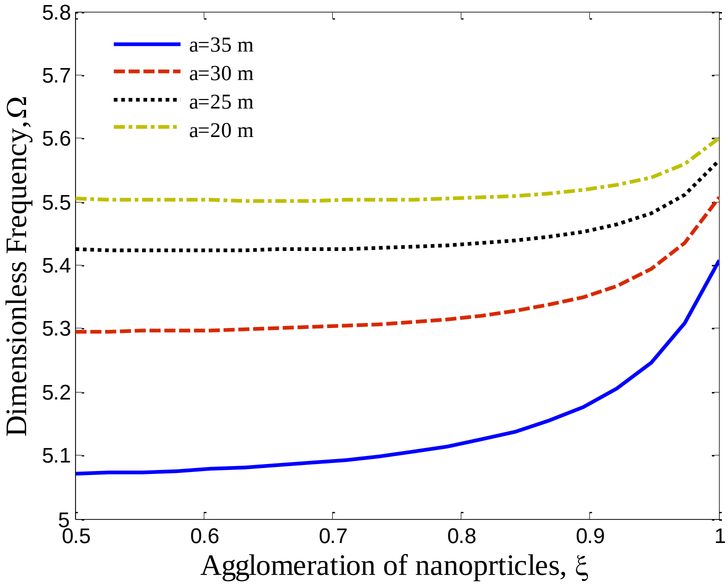

Fig. 4 Effects of length of concrete foundation on the dimensionless frequencyversus agglomeration percent

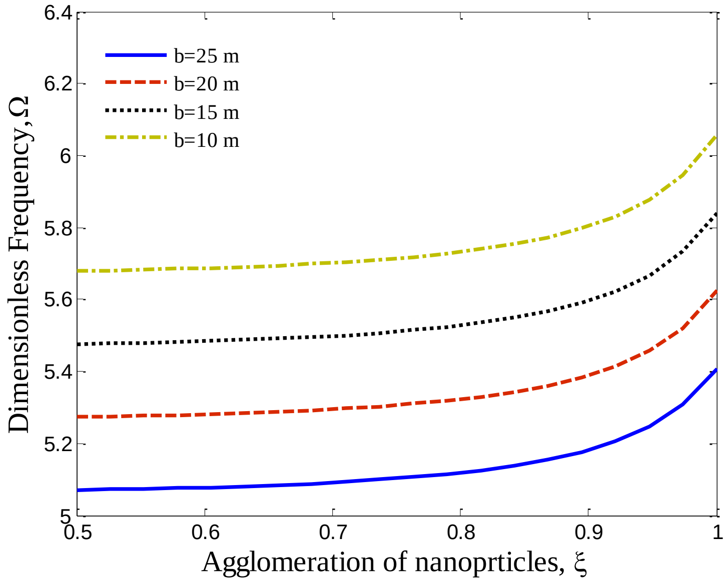

Fig. 5 Effects of width of concrete foundation on the dimensionless frequency versus agglomeration percent

structure. Furthermore, the frequency of the dense sand medium is higher than other cases since the spring constant of this medium is maximum.

The effect of the length of concrete foundation on the dimensionless frequency of the system versus agglomeration percent of SiO_2 nanoparticles is depicted in Fig. 4. As can be seen, the frequency of the structure decreases with increasing the length of concrete foundation. It is because increasing the length leads to softer structure.

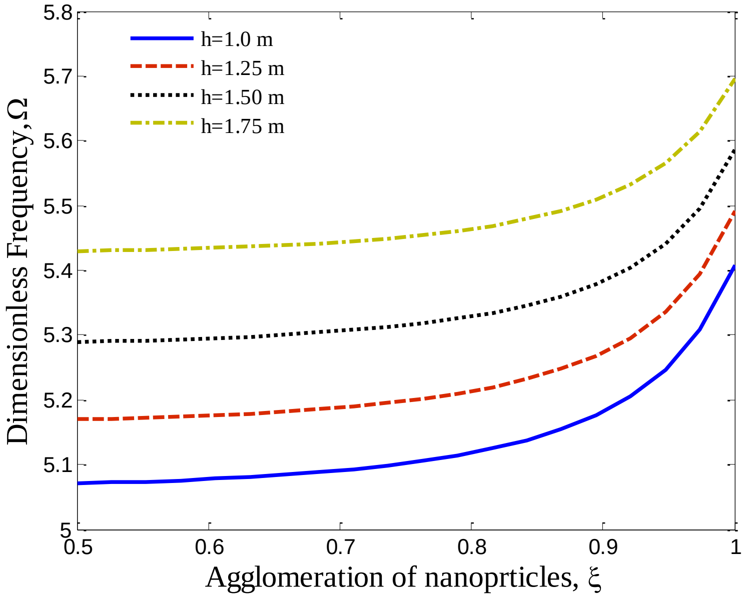

Fig. 6 Effects of thickness of concrete foundation on the dimensionless frequency versus agglomeration percent

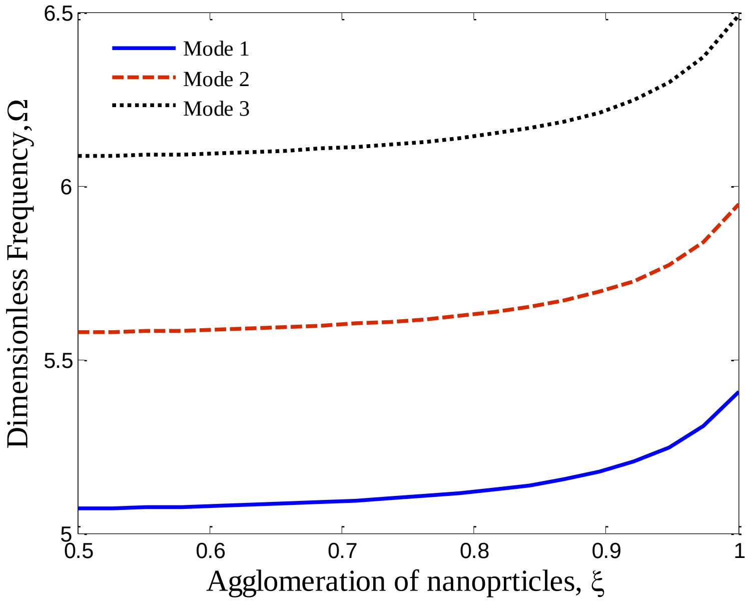

Fig. 7 Effects of mode number on the dimensionless frequency versus agglomeration percent

Fig. 5 shows the dimensionless frequency of the structure versus agglomeration percent of SiO_2 nanoparticles for different width the concrete foundation. It can be also found that the frequency of the structure decrease with increasing the width which is due to the higher stiffness of system with lower width of concrete foundation.

The effect of thickness of concrete foundation on the dimensionless frequency versus agglomeration percent of SiO_2 nanoparticles is shown in Fig. 6. It can be found that with increasing the thickness, the frequency of the structure is increased. It is because with increasing the thickness, the stiffness of the structure will be improved.

The effect of mode numbers on the dimensionless frequency versus agglomeration percent of SiO_2 nanoparticles of system is plotted in Fig. 7. As can be seen, with increasing the mode numbers, the frequency increases.

5. Conclusions

Vibration of concrete foundations reinforced with SiO_2 nanoparticles resting on soil medium was the main contribution of the present paper. Mori-Tanaka model was used for obtaining the effective material properties of the structure considering agglomeration effects. The soil medium was simulated by Winkler foundation. Based on orthotropic Reddy theory, the motion equations were derived using energy method and Hamilton's principle. Exact solution is applied for obtaining the frequency of system so that the effects of the volume percent and agglomeration of SiO_2 nanoparticles, soil medium and geometrical parameters of concrete foundation were considered. It can be seen that with increasing the values of SiO_2 nanoparticles volume fraction, the frequency of the system was increased. Considering agglomeration of SiO_2 nanoparticles leads to lower frequency. It can be seen that considering soil medium increases the frequency of the structure. Furthermore, the frequency of the dense sand medium was higher than other cases since the spring constant of this medium was maximum. In addition, the frequency of the structure decreases with increasing the length to thickness ratio and length to width ratio of the concrete foundation. Present results are in good agreement with those reported by the other references. Finally, it is hoped that the results presented in this paper would be helpful for design of concrete foundations.

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CC

Appendix A

$$K_{11} = -\frac{Q_{11} m^2 \pi^2 h}{L^2} - \frac{Q_{66} n^2 \pi^2 h}{b^2} \quad \text{(A1)}$$

$$K_{12} = -\frac{Q_{12} m \pi^2 n h}{L b} - \frac{Q_{66} m \pi^2 n h}{L b} \quad \text{(A2)}$$

$$K_{13} = 0 \quad \text{(A3)}$$

$$K_{14} = 0 \quad \text{(A4)}$$

$$K_{15} = 0 \quad \text{(A5)}$$

$$K_{21} = -\frac{Q_{12} m \pi^2 n h}{L b} - \frac{Q_{66} m \pi^2 n h}{L b} \quad \text{(A6)}$$

$$K_{22} = -\frac{Q_{66} m^2 \pi^2 h}{L^2} - \frac{Q_{22} n^2 \pi^2 h}{b^2} \quad \text{(A7)}$$

$$K_{23} = 0 \quad \text{(A8)}$$

$$K_{24} = 0 \quad \text{(A9)}$$

$$K_{25} = 0 \quad \text{(A10)}$$

$$K_{31} = 0 \quad \text{(A11)}$$

$$K_{32} = 0 \quad \text{(A12)}$$

$$\begin{aligned} K_{33} = & -\frac{Q_{55} m^2 \pi^2 h}{L^2} - c2\left(\frac{3}{80}\frac{Q_{55}\, c1\, m^2 \pi^2 h^5}{L^2} \right. \\ & \left. - \frac{1}{12}\frac{Q_{55} m^2 \pi^2 h^3}{L^2}\right) + \frac{1}{4}\frac{Q_{44}\, c1\, n^2 \pi^2 h^3}{b^2} \\ & - \frac{Q_{44} n^2 \pi^2 h}{b^2} - c2\left(\frac{3}{80}\frac{Q_{44}\, c1\, n^2 \pi^2 h^5}{b^2} \right. \\ & \left. - \frac{1}{12}\frac{Q_{44} n^2 \pi^2 h^3}{b^2}\right) - \frac{N_{xm} m^2 \pi^2}{L^2} \end{aligned}$$

$$
- \frac{N_{ym}\, n^2\, \pi^2}{b^2} + \frac{1}{4}\, \frac{Q_{55}\, c1\, m^2\, \pi^2\, h^3}{L^2} + c1 \Bigg(- \frac{1}{112}\, \frac{Q_{66}\, c1\, m^2\, \pi^4\, n^2\, h^7}{L^2\, b^2} + \frac{1}{448} \Bigg(- \frac{Q_{11}\, c1\, m^4\, \pi^4}{L^4} - \frac{Q_{12}\, c1\, m^2\, \pi^4\, n^2}{L^2\, b^2} \Bigg) h^7 \Bigg) - K_w \tag{A13}
$$

$$
K_{34} = -c2 \left(\frac{3}{80}\, \frac{Q_{55}\, c1\, m\, \pi\, h^5}{L} - \frac{1}{12}\, \frac{Q_{55}\, m\, \pi\, h^3}{L} \right) + c1 \Bigg(- \frac{1}{448}\, \frac{Q_{11}\, c1\, m^3\, \pi^3\, h^7}{L^3} + \frac{1}{40}\, \frac{Q_{66}\, m\, \pi^3\, n^2\, h^5}{L\, b^2} + \frac{1}{80}\, \frac{Q_{11}\, m^3\, \pi^3\, h^5}{L^3} - \frac{1}{224}\, \frac{Q_{66}\, c1\, m\, \pi^3\, n^2\, h^7}{L\, b^2} \Bigg) - \frac{Q_{55}\, m\, \pi\, h}{L} + \frac{1}{4}\, \frac{Q_{55}\, c1\, m\, \pi\, h^3}{L} \tag{A14}
$$

$$
K_{35} = c1 \Bigg(- \frac{1}{448}\, \frac{Q_{12}\, c1\, m^2\, \pi^3\, n\, h^7}{L^2\, b} + \frac{1}{40}\, \frac{Q_{66}\, m^2\, \pi^3\, n\, h^5}{L^2\, b} + \frac{1}{80}\, \frac{Q_{12}\, m^2\, \pi^3\, n\, h^5}{L^2\, b} - \frac{1}{224}\, \frac{Q_{66}\, c1\, m^2\, \pi^3\, n\, h^7}{L^2\, b} \Bigg) + \frac{1}{4}\, \frac{Q_{44}\, c1\, n\, \pi\, h^3}{b} - \frac{Q_{44}\, n\, \pi\, h}{b} - c2 \left(\frac{3}{80}\, \frac{Q_{44}\, c1\, n\, \pi\, h^5}{b} - \frac{1}{12}\, \frac{Q_{44}\, n\, \pi\, h^3}{b} \right) \tag{A15}
$$

$$
K_{41} = 0 \tag{A16}
$$

$$
K_{42} = 0 \tag{A17}
$$

$$
\begin{aligned}
K_{43} = & -\frac{1}{224}\,\frac{c2\,Q_{66}\,c1\,m\,\pi^3\,n^2\,h^7}{L\,b^2} \\
& + \frac{1}{40}\,\frac{Q_{66}\,c1\,m\,\pi^3\,n^2\,h^5}{L\,b^2} \\
& + \frac{1}{4}\,\frac{Q_{55}\,c1\,m\,\pi\,h^3}{L} \\
& - \frac{1}{448}\,c1\left(\frac{Q_{11}\,c1\,m^3\,\pi^3}{L^3}\right. \\
& \left. + \frac{Q_{12}\,c1\,m\,\pi^3\,n^2}{L\,b^2}\right)h^7 - \frac{Q_{55}\,m\,\pi\,h}{L} \\
& + \frac{1}{80}\left(\frac{Q_{11}\,c1\,m^3\,\pi^3}{L^3} + \frac{Q_{12}\,c1\,m\,\pi^3\,n^2}{L\,b^2}\right)h^5 \\
& + c2\left(-\frac{3}{80}\,\frac{Q_{55}\,c1\,m\,\pi\,h^5}{L}\right. \\
& \left. + \frac{1}{12}\,\frac{Q_{55}\,m\,\pi\,h^3}{L}\right)
\end{aligned}
\tag{A18}
$$

$$
\begin{aligned}
K_{44} = & \, c2\left(-\frac{3}{80}\,Q_{55}\,c1\,h^5 + \frac{1}{12}\,Q_{55}\,h^3\right) \\
& - c1\left(\frac{1}{448}\,\frac{Q_{11}\,c1\,m^2\,\pi^2\,h^7}{L^2}\right. \\
& \left. - \frac{1}{80}\,\frac{Q_{11}\,m^2\,\pi^2\,h^5}{L^2}\right) - \frac{1}{12}\,\frac{Q_{11}\,m^2\,\pi^2\,h^3}{L^2} \\
& + \frac{1}{80}\,\frac{Q_{11}\,c1\,m^2\,\pi^2\,h^5}{L^2} \\
& + \frac{1}{80}\,\frac{Q_{66}\,c1\,n^2\,\pi^2\,h^5}{b^2} + \frac{1}{4}\,Q_{55}\,c1\,h^3 \\
& - c2\left(\frac{1}{448}\,\frac{Q_{66}\,c1\,n^2\,\pi^2\,h^7}{b^2}\right. \\
& \left. - \frac{1}{80}\,\frac{Q_{66}\,n^2\,\pi^2\,h^5}{b^2}\right) - \frac{1}{12}\,\frac{Q_{66}\,n^2\,\pi^2\,h^3}{b^2} \\
& - Q_{55}\,h
\end{aligned}
\tag{A19}
$$

$$
\begin{aligned}
K_{45} = {} & \frac{1}{80}\,\frac{Q_{12}\,c1\,m\,\pi^2\,n\,h^5}{L\,b} \\
& - c1\left(\frac{1}{448}\,\frac{Q_{12}\,c1\,m\,\pi^2\,n\,h^7}{L\,b}\right. \\
& \left. - \frac{1}{80}\,\frac{Q_{12}\,m\,\pi^2\,n\,h^5}{L\,b}\right) - \frac{1}{12}\,\frac{Q_{12}\,m\,\pi^2\,n\,h^3}{L\,b} \\
& - \frac{1}{12}\,\frac{Q_{66}\,m\,\pi^2\,n\,h^3}{L\,b} \\
& + \frac{1}{80}\,\frac{Q_{66}\,c1\,m\,\pi^2\,n\,h^5}{L\,b} \\
& - c2\left(\frac{1}{448}\,\frac{Q_{66}\,c1\,m\,\pi^2\,n\,h^7}{L\,b}\right. \\
& \left. - \frac{1}{80}\,\frac{Q_{66}\,m\,\pi^2\,n\,h^5}{L\,b}\right)
\end{aligned} \tag{A20}
$$

$$
K_{51} = 0 \tag{A21}
$$

$$
K_{52} = 0 \tag{A22}
$$

$$
\begin{aligned}
K_{53} = {} & \frac{1}{40}\,\frac{Q_{66}\,c1\,m^2\,\pi^3\,n\,h^5}{L^2\,b} + \frac{1}{4}\,\frac{Q_{44}\,c1\,n\,\pi\,h^3}{b} \\
& + \frac{1}{80}\left(\frac{Q_{12}\,c1\,m^2\,\pi^3\,n}{L^2\,b} + \frac{Q_{22}\,c1\,n^3\,\pi^3}{b^3}\right)h^5 \\
& - \frac{Q_{44}\,n\,\pi\,h}{b} - \frac{1}{224}\,\frac{c1^2\,Q_{66}\,m^2\,\pi^3\,n\,h^7}{L^2\,b} \\
& + c2\left(-\frac{3}{80}\,\frac{Q_{44}\,c1\,n\,\pi\,h^5}{b}\right. \\
& \left. + \frac{1}{12}\,\frac{Q_{44}\,n\,\pi\,h^3}{b}\right)
\end{aligned} \tag{A23}
$$

$$
\begin{aligned}
K_{54} = {} & \frac{1}{80}\,\frac{Q_{12}\,c1\,m\,\pi^2\,n\,h^5}{L\,b} \\
& + \frac{1}{80}\,\frac{Q_{66}\,c1\,m\,\pi^2\,n\,h^5}{L\,b} \\
& - c1\left(\frac{1}{448}\,\frac{Q_{66}\,c1\,m\,\pi^2\,n\,h^7}{L\,b}\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{80}\,\frac{Q_{66}\,m\,\pi^{2}\,n\,h^{5}}{L\,b}\Bigg) - \frac{1}{12}\,\frac{Q_{66}\,m\,\pi^{2}\,n\,h^{3}}{L\,b} \\
&- \frac{1}{12}\,\frac{Q_{12}\,m\,\pi^{2}\,n\,h^{3}}{L\,b}
\end{aligned}
\tag{A24}
$$

$$
\begin{aligned}
K_{55} = {} & \frac{1}{80}\,\frac{Q_{66}\,c1\,m^{2}\,\pi^{2}\,h^{5}}{L^{2}} - Q_{44}\,h \\
& + \frac{1}{80}\,\frac{Q_{22}\,c1\,n^{2}\,\pi^{2}\,h^{5}}{b^{2}} - \frac{1}{12}\,\frac{Q_{66}\,m^{2}\,\pi^{2}\,h^{3}}{L^{2}} \\
& - c1\Bigg(\frac{1}{448}\,\frac{Q_{66}\,c1\,m^{2}\,\pi^{2}\,h^{7}}{L^{2}} \\
& - \frac{1}{80}\,\frac{Q_{66}\,m^{2}\,\pi^{2}\,h^{5}}{L^{2}}\Bigg) + \frac{1}{4}\,Q_{44}\,c1\,h^{3} \\
& - \frac{1}{12}\,\frac{Q_{22}\,n^{2}\,\pi^{2}\,h^{3}}{b^{2}} + c2\Bigg(-\frac{3}{80}\,Q_{44}\,c1\,h^{5} \\
& + \frac{1}{12}\,Q_{44}\,h^{3}\Bigg)
\end{aligned}
\tag{A25}
$$