The 2019 World Congress on
Advances in Structural Engineering and Mechanics (ASEM19)
Jeju Island, Korea, September 17 - 21, 2019

Keynote Paper

Topology Optimization of Buildings Structures subject to Stochastic Dynamic Loads

Fernando Gomez[1)] and *Billie F. Spencer, Jr.[2)]

[1), 2)] *Department of Civil and Environmental Engineering, University of Illinois at Urbana-Champaign, 205 North Mathews Avenue, MC-250, Urbana, IL 61801, USA*
[2)] *bfs@illinois.edu*

ABSTRACT

Topology optimization provides a general approach to obtain optimal material layout in a prescribed domain according to some cost function and subjected to given design constraints; however, most approaches only accommodate deterministic loads although many of the most severe dynamic loads that civil structures withstand are stochastic in nature. In contrast, this study accounts directly the stochastic excitation, by modeling it as a zero-mean filtered white noise; when combined with the equations of motion for the structure, an augmented state space representation is formed, whose only input is an uncorrelated white noise; and the stationary covariances of the structural responses of interest are obtained by solving a large-scale Lyapunov equation. The optimization problem is formulated with a general objective function defined in terms of the covariances of the structural responses, a volumetric constraint, and design variables as the relative densities of the elements. A gradient-based method is used for the update of the design variables, and the sensitivities are computed using an efficient adjoint method that requires the solution of an adjoint Lyapunov equation. Additionally, this study implements the following details in the topology optimization of stochastically excited buildings: additional floor masses, gravity boundary elements, diaphragm constraints, and ground motion stochastic models. To illustrate the framework, topology optimization of the lateral resisting system of a mid-rise building under lateral seismic excitation is performed. The results show the opportunities of topology optimization of stochastically excited structures.

1. INTRODUCTION

Current structural design procedures are based on an iterative process, which guarantees structural safety but not optimal economy (Xu, et al. 2017). In this regard, topology optimization provides a general approach to obtain optimal material layout in a prescribed domain according to some cost function and subjected to given design

---

[1)] Doctoral Candidate
[2)] Nathan M. and Anne M. Newmark Endowed Chair in Civil Engineering

constraints (Bendsøe and Sigmund 2003). Extensive research has been conducted in this field to develop well-posed formulations (Bendsøe and Kikuchi 1988, Sigmund and Petersson 1998, Sigmund 2007) and to solve the numerical problems generated by this approach, such as mesh dependency, checkerboarding, islanding, and local minima (Diaz and Sigmund 1995, Sigmund and Petersson 1998).

Topology optimization has been successfully applied to solve the minimum compliance problem subjected to deterministic static loading for general structures (Bendsøe and Sigmund 2003, Talischi, et al. 2012), as well as buildings (Stromberg, et al. 2012). It also has been applied to dynamic problems such as eigen-frequency optimization for free vibration (Olhoff 1989) and minimum dynamic compliance for forced harmonic vibration (Ma, et al. 1995). However, such deterministic approaches cannot accommodate stochastic dynamic loads which civil structures frequently undergo (e. g., winds, earthquakes, traffic, etc.; see Soong and Grigoriu 1993), and therefore, produce suboptimal designs. Recently, Gomez and Spencer (2019a) proposed a framework for topology optimization of general structures subjected to stochastic dynamic loading, in which a large-scale Lyapunov equation was solved to obtain the covariance of the response. This method provided promising results in topology optimization of stochastically excited structures, and it is the base for the present study. In addition, in buildings only the response of few nodes is of interest, therefore a model reduction technique can be applied to reduce the order of the system to the size of the points of interest.

This study implements a topology optimization scheme for stochastically excited structures, using a performance function in terms of the covariance of the stationary structural responses, obtained by solving a large-scale Lyapunov equation. An adjoint method is used to obtain the sensitivities of the performance function, which allows the use of efficient gradient-based updating procedures. Illustrative examples are provided for the optimization of the lateral resisting system of a building subject to stochastic dynamic excitation. The results demonstrate the efficacy of the proposed approach for multi-objective topology optimization of stochastically excited structures.

2. STOCHASTIC EXCITATION AND RESPONSE

This section formulates the equations of motion as a state space representation and models the input excitation as a filtered white noise. An augmented state space representation is then formed, and the stationary covariance of the responses are obtained via solution of the Lyapunov equation.

2.1 Equation of Motion and State Space Representation

The equation of motion (EOM) of an arbitrary N-degree-of-freedom linear system is given by

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{G}\mathbf{f}(t) \tag{1}$$

where $\mathbf{M}$ is the mass matrix; $\mathbf{C}$ is the damping matrix; $\mathbf{K}$ is the stiffness matrix; $\mathbf{u}$ is the displacement vector; $\mathbf{f}(t)$ is the input excitation vector; and $\mathbf{G}$ is the load-effect matrix. In topology optimization, these matrices are typically obtained using a Galerkin finite element approximation with first order shape functions. The input excitation vector, $\mathbf{f}(t)$ is assumed to be a stationary stochastic process.

Defining the state vector as

$$\mathbf{x} = \begin{bmatrix} \mathbf{u}^{\mathrm{T}} & \dot{\mathbf{u}}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tag{2}$$

the state space representation of the system is given by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_{\mathrm{s}}\mathbf{x} + \mathbf{B}_{\mathrm{s}}\mathbf{f}(t) \\ \mathbf{y} &= \mathbf{C}_{\mathrm{s}}\mathbf{x} + \mathbf{D}_{\mathrm{s}}\mathbf{f}(t) \end{aligned} \tag{3}$$

where the matrices $\mathbf{A}_{\mathrm{s}}$ and $\mathbf{B}_{\mathrm{s}}$ are given by

$$\mathbf{A}_{\mathrm{s}} = \begin{bmatrix} \mathbf{0}_{N\times N} & \mathbf{I}_{N\times N} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \quad \mathbf{B}_{\mathrm{s}} = \begin{bmatrix} \mathbf{0}_{N\times 1} \\ \mathbf{M}^{-1}\mathbf{G} \end{bmatrix} \tag{4}$$

and the matrices $\mathbf{C}_{\mathrm{s}}$ and $\mathbf{D}_{\mathrm{s}}$ depend on the desired output $\mathbf{y}$.

2.2 Stochastic Excitation Model

In this study, the excitation is assumed as a zero-mean stationary stochastic process that can be modeled as a filtered white noise, admitting the following state space representation

$$\begin{aligned} \dot{\mathbf{x}}_{\mathrm{g}} &= \mathbf{A}_{\mathrm{g}}\mathbf{x}_{\mathrm{g}} + \mathbf{B}_{\mathrm{g}}\mathbf{w}(t) \\ \mathbf{f} &= \mathbf{C}_{\mathrm{g}}\mathbf{x}_{\mathrm{g}} \end{aligned} \tag{5}$$

where $\mathbf{x}_{\mathrm{g}}$ is the state vector of the excitation; the matrices $\mathbf{A}_{\mathrm{g}}$, $\mathbf{B}_{\mathrm{g}}$, and $\mathbf{C}_{\mathrm{g}}$ are determined by the characteristics of the excitation; and $\mathbf{w}(t)$ is a multi-dimensional white noise described by

$$\mathbb{E}(\mathbf{w}(t)) = 0, \quad \mathbb{E}(\mathbf{w}(t_1)\mathbf{w}(t_2)) = 2\pi\mathbf{S}_0\delta(t_1 - t_2) \tag{6}$$

where $\mathbb{E}(\cdot)$ is the expected value operator; $\mathbf{S}_0$ as the constant two-sided power spectral density matrix of the noise; and $\delta(\cdot)$ is the Dirac delta function.

Defining the augmented state vector as

$$\mathbf{x}_{\mathrm{a}} = \begin{bmatrix} \mathbf{x}^{\mathrm{T}} & \mathbf{x}_{\mathrm{g}}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tag{7}$$

then the augmented system is given as

$$\begin{aligned} \dot{\mathbf{x}}_{\mathrm{a}} &= \mathbf{A}_{\mathrm{a}}\mathbf{x}_{\mathrm{a}} + \mathbf{B}_{\mathrm{a}}\mathbf{w}(t) \\ \mathbf{y} &= \mathbf{C}_{\mathrm{a}}\mathbf{x}_{\mathrm{a}} \end{aligned} \tag{8}$$

where the matrices $\mathbf{A}_\mathrm{a}$, $\mathbf{B}_\mathrm{a}$, and $\mathbf{C}_\mathrm{a}$ are

$$\mathbf{A}_\mathrm{a} = \begin{bmatrix} \mathbf{A}_\mathrm{s} & \mathbf{B}_\mathrm{s}\mathbf{C}_\mathrm{g} \\ \mathbf{0}_{N_g \times N} & \mathbf{A}_\mathrm{g} \end{bmatrix}, \quad \mathbf{B}_\mathrm{a} = \begin{bmatrix} \mathbf{0}_{2N\times 1} \\ \mathbf{B}_\mathrm{g} \end{bmatrix}, \quad \mathbf{C}_\mathrm{a} = \begin{bmatrix} \mathbf{D}_\mathrm{s}\mathbf{C}_\mathrm{g} & \mathbf{C}_\mathrm{s} \end{bmatrix}. \tag{9}$$

2.3 Stochastic Structural Response

The covariance matrix of the vector $\mathbf{x}_\mathrm{a}$ is defined as

$$\mathbf{\Gamma}_{\mathbf{x}_\mathrm{a}} = \mathbb{E}((\mathbf{x}_\mathrm{a} - \mu_{\mathbf{x}_\mathrm{a}})(\mathbf{x}_\mathrm{a} - \mu_{\mathbf{x}_\mathrm{a}})^\mathrm{T}) \tag{10}$$

Because the excitation is a zero-mean process, then the response is also a zero-mean process; consequently, the covariance of the response is given by

$$\mathbf{\Gamma}_{\mathbf{x}_\mathrm{a}} = \mathbb{E}(\mathbf{x}_\mathrm{a}\mathbf{x}_\mathrm{a}^\mathrm{T}) \tag{11}$$

The covariance of the stationary response of the augmented state is governed by the Lyapunov equation

$$\mathbf{A}_\mathrm{a}\mathbf{\Gamma}_{\mathbf{x}_\mathrm{a}} + \mathbf{\Gamma}_{\mathbf{x}_\mathrm{a}}\mathbf{A}_\mathrm{a}^\mathrm{T} + 2\pi\mathbf{B}_\mathrm{a}\mathbf{S}_0\mathbf{B}_\mathrm{a}^\mathrm{T} = \mathbf{0} \tag{12}$$

and the covariance of the system output $\mathbf{y}$ is given by

$$\mathbf{\Gamma}_\mathbf{y} = \mathbb{E}(\mathbf{y}\mathbf{y}^\mathbf{T}) = \mathbf{C}_\mathrm{a}\mathbf{\Gamma}_{\mathbf{x}_\mathrm{a}}\mathbf{C}_\mathrm{a}^\mathrm{T}. \tag{13}$$

Because the matrix $\mathbf{A}_\mathrm{a}$ is Hurwitz, and the matrix $2\pi\mathbf{B}_\mathrm{a}\mathbf{S}_0\mathbf{B}_\mathrm{a}^\mathrm{T}$ given in the last term of Eq. (12) is positive semidefinite, the solution for $\mathbf{\Gamma}_\mathbf{y}$ is unique and positive semidefinite.

3. PROBLEM FORMULATION AND SOLUTION

This section summarizes the topology optimization framework for structures subjected to stochastic excitations proposed by Gomez and Spencer (2019a), including the objective function and the constraints. Some details of the topology optimization solution process are also presented.

3.1 Building Characteristics

Typical buildings differ from other types of structures due to some unique features that will be described next, as shown in Fig. 1. The topology optimization framework proposed in this study is tailored to consider these characteristics in order to provide more realistic examples. One of the most important characteristics is that building are divided into floors not necessarily of the same height, and only the response of the floors is of interest to the designer. In addition, each floor contains structural elements such as slabs and floor framing that are not part of the lateral resisting system (LRS) and non-structural elements such as finishes, partition walls, and permanent live loads.

All these elements provide additional structural masses to the systems and are not included in typical examples of topology optimization of buildings. The additional masses can be modeled as lumped masses in some floor nodes or distributed among all nodes in the floor. These features allow a model reduction to improve the efficiency of the solution (Gomez and Spencer 2019b).

Another feature is that besides the LRS, the structure has a gravitational resisting system, and some elements belong to both, especially in the boundary between both systems. For example, there exist boundary columns in the LRS, which supports gravitational and lateral loads. Another important detail of buildings is that the slab to support gravitational loads, also acts as a diaphragm to redistribute lateral loads. The constraint can be enforced using a transformation matrix in this type of problems (Gomez and Spencer 2019a).

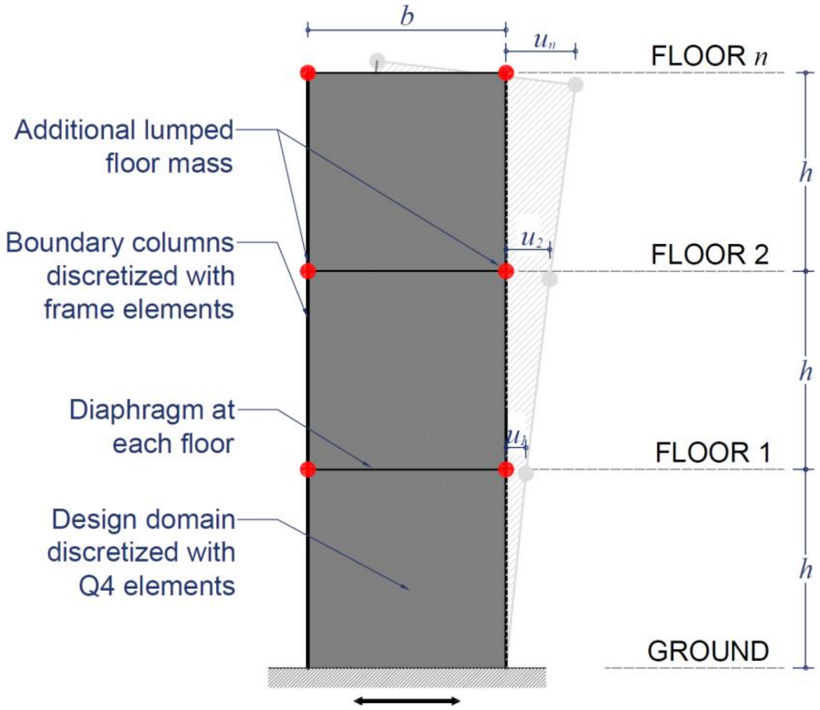


Fig. 1 Characteristics of a building subjected to stochastic dynamic excitation

3.2 Topology Optimization Formulation

The topology optimization is based on intermediate element densities (Bendsøe and Sigmund 2003), such that for each element, a relative density variable $\boldsymbol{z}$ is chosen. SIMP interpolation is used for the Young's modulus and density for each element, which yields

$$\begin{aligned} E(z) &= [\epsilon + (1-\epsilon) z^p] E^0 \\ \rho(z) &= [\epsilon + (1-\epsilon) z^q] \rho^0 \end{aligned} \tag{14}$$

where E^0 and ρ^0 are the Young's modulus and density for the solid material, p and q are the penalization factors, and ϵ is a small number.

The optimization formulation is given by finding the vector of density variables **z**,

$$\begin{aligned} \min_{\mathbf{z}} \quad & J(\mathbf{z}) = \max_{i\in S}\{J_i(\mathbf{z})\} = \max_{i\in S}\{\mathbf{F}_i(\mathbf{z}) : \mathbf{\Gamma}_{\mathbf{x}_\mathrm{a}}(\mathbf{z})\} \\ \mathrm{s.t.} \quad & g(\mathbf{z}) = V(\mathbf{z}) - V_{\max} \leq 0 \\ & \mathbf{A}_\mathrm{a}\mathbf{\Gamma}_{\mathbf{x}_\mathrm{a}} + \mathbf{\Gamma}_{\mathbf{x}_\mathrm{a}}\mathbf{A}_\mathrm{a}^\mathrm{T} + 2\pi\mathbf{B}_\mathrm{a}\mathbf{S}_0\mathbf{B}_\mathrm{a}^\mathrm{T} = \mathbf{0} \\ & z_n \in [z_{\min}, z_{\max}] \text{ for } n = 1,2,\dots,N_\mathrm{el} \end{aligned} \tag{15}$$

where : represents the double dot product between matrices, S is a set of indices, $\mathbf{F}_i$ is a symmetric positive semidefinite matrix, V is the volume, $V_{\max}$ is the volume limit, and $z_{\min}$ and $z_{\max}$ are the lower and upper bounds on the density variables. Note that the proposed performance function allows the envelope of many types of responses (e.g., displacements, drifts, accelerations) of one or many points; and that the performance function is completely defined by the covariance of the response. For example, minimizing the maximum response among all the floors is possible.

As indicated previously, the covariance of the response of the structure subjected to a stochastic process is obtained by solving the Lyapunov equation in Eq. (12), and consequently, the topology optimization problem is deterministic.

3.3 Optimization Details

The solution of the proposed optimization problem is summarized in Fig. 2. In the initialization step, the domain is meshed, the element matrices using solid material are computed, the matrices for the excitation model are constructed, and the initial values for the design variables are chosen. Additionally, to avoid mesh-dependency and numerical instabilities such as checkerboard patterns and islanding, a filter is applied to the sensitivities (Sigmund and Petersson 1998). A linear hat filter is implemented through a filter matrix that is computed in the initialization step (Talischi et al. 2012).

The remainder of the steps follow an iterative procedure. In the analysis step, the system matrices are obtained using the current values for the design variables, a model reduction is applied to obtain a reduced system, and then, the covariance of the response is computed by solving the reduced Lyapunov equation (Gomez and Spencer 2019b). In the sensitivity step, the adjoint Lyapunov equations are solved to obtain the Lagrange multiplier and the performance function and constraints sensitivities (Gomez and Spencer 2019b). In the update step, the new values for the design variables are obtained by using the method of moving asymptotes (Svanberg 1987). The iterative scheme is applied until the maximum change in the design variables is below a specified threshold.

4. ILLUSTRATIVE EXAMPLES

In this section, the proposed framework is illustrated by optimizing the lateral force resisting system of a 9-story building subjected to a stochastic ground motion (Ohtori et al. 2004). In the equation of motion, **G** is the load-effect vector, which has components equal to 1 only for the DOFs in the direction of the ground motion, **u** is the relative displacement to the ground, and f is the scalar stochastic ground acceleration.

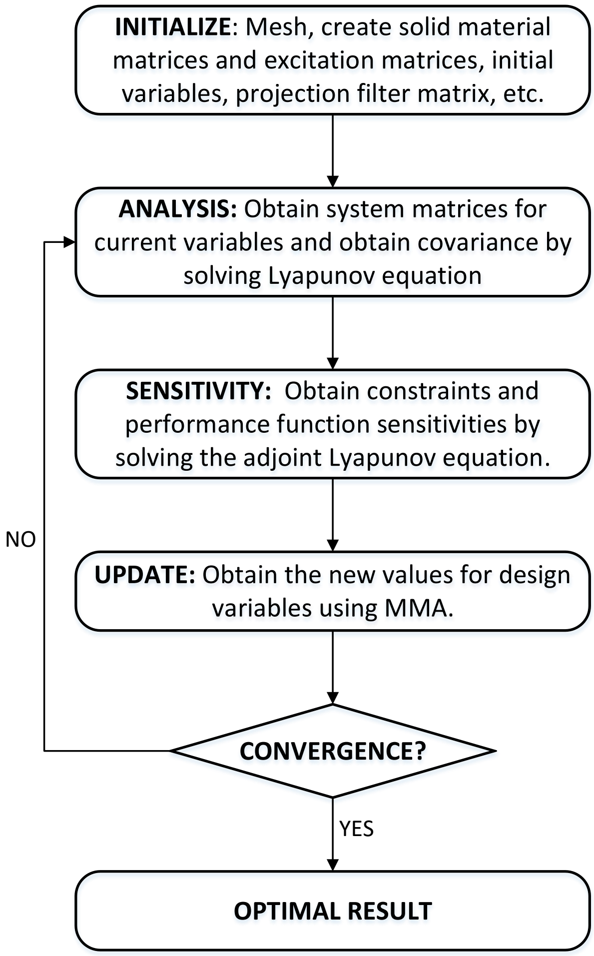


Fig. 2 Topology optimization flowchart for stochastically excited structures

The design domain is given by the $45\mathrm{m} \times 36\mathrm{m}$ rectangle, which is composed of a solid linear elastic material having the following properties, which are representative of structural steel: Young's modulus $E^0 = 200\,\mathrm{GPa}$, Poisson's ratio $\nu = 0.3$, density $\rho^0 = 7500\,\frac{kg}{m^3}$, and Ersatz parameter $\epsilon = 10^{-4}$. The domain has a uniform thickness of 0.05 m; and due to its thickness, the continuum domain is assumed to be in plane stress condition. The continuum domain is discretized using 180×144 Q4 elements. The building has 9 floors with the same height equal to 4 m and 5 bays with the same span equal to 9 m. The spans are divided by columns with section that vary linearly from W14x500 to W14x257, the columns are discretized using 864 frame elements with axial and bending stiffness; the material properties are the same as the design domain. In addition, lumped masses of 85 Ton are located in each floor and in each axis; these masses represent the structural and non-structural elements not included in the model. Pinned supports are applied to the column bases. Two types of axial stiffness are considered in all floors: rigid diaphragm constraints such that the nodes in each floor have equal lateral displacements, and a flexible diaphragm with axial stiffness provided by floor beams W24x131. The radius of the filter is equal to 0.40. The volume of the optimization variables is constrained to be less or equal than 0.20 of the solid domain. The damping matrix is obtained using Rayleigh damping with 2% damping ratio for the first two modes. The objective is to minimize the maximum interstory drift among all stories.

The ground motion $f(t)$ is modeled as a Clough-Penzien model with $\omega_{\mathrm{g}} = 15\,\mathrm{rad/\,s}$, $\zeta_{\mathrm{g}} = 0.60$, $\omega_{\mathrm{f}} = 1.5\,\mathrm{rad/s}$, $\zeta_{\mathrm{g}} = 0.60$, and $S_0 = 0.026\mathrm{m}^2/\mathrm{s}^3$, whose power spectral density is shown in Fig. 3; these parameters are typical of stiff soil (Clough and Penzien 1993) with an approximate peak ground acceleration of 0.5 times gravity.

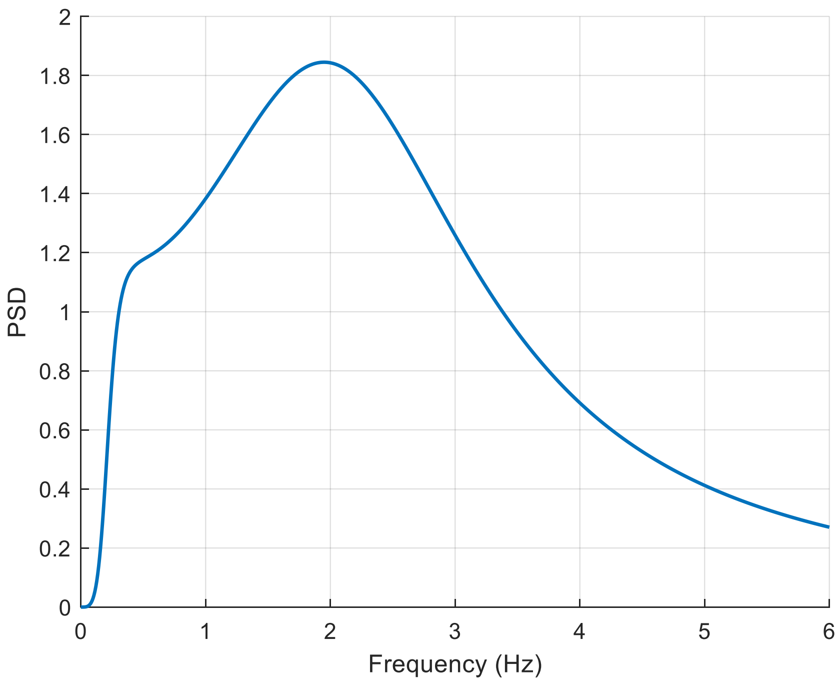
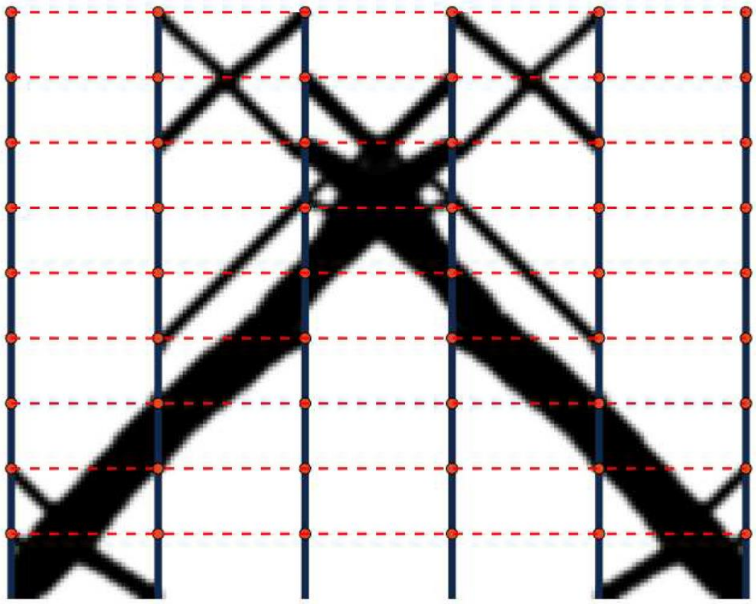


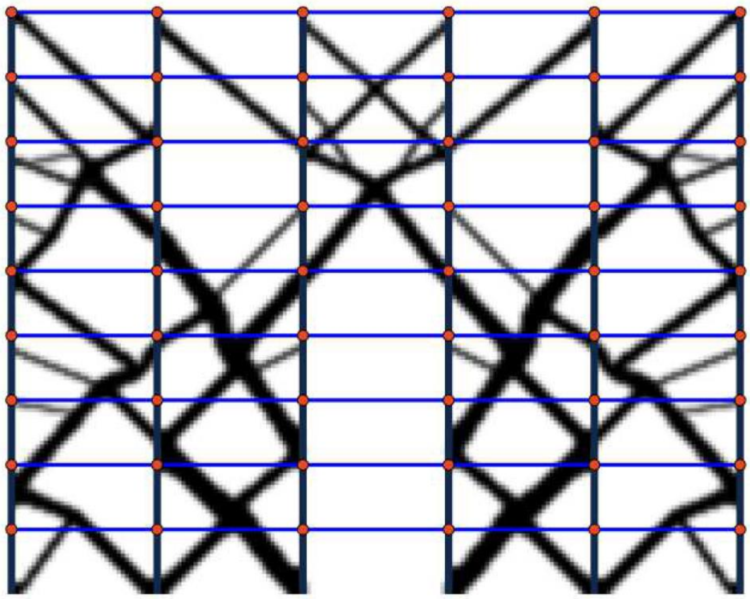
Fig. 3 Power spectral density of ground motion excitation

Fig. 4a-b show the optimal design for both types of diaphragms; the first natural frequencies are equal to 2.45 and 4.13 Hz for the case with rigid diaphragms and to 1.99 and 2.30 Hz for the case with flexible diaphragms. These figures show that the optimal topology heavily depends on the type of diaphragm. In the case with rigid diaphragms, there exists 2 super braces spanning 7 floors and additional sets of smaller braces spanning 2 floors. In the case with flexible diaphragms, there exists a short beam with bracing patterns spanning 6 floors and additional braces towards the other floors with a belt truss spanning the two upper floors.

The maximum interstory drift variance in both designs are 0.253×10^{-4} and 0.498×10^{-4}, respectively. It is worth noting that in the rigid diaphragms case, the interstory drifts variance in all floors are equal to the maximum interstory drift variance, i.e., there is a uniform distribution of interstory drift variance, which means there would be a uniform distribution of damage in all floors during a seismic event. Moreover, in the case of flexible diaphragms, the interstory drift distribution in all axes in all floors is also uniform, which is achieved due to the resulting topology.



(a)



(b)

Fig. 4 Optimized topologies for minimizing maximum interstory drift applying (a) rigid and (b) flexible diaphragms in each floor

5. CONCLUSIONS

This paper implemented a multi-objective framework for topology optimization of stochastically excited structures. The input was modeled as a filtered white noise, and the performance of the structure due to this excitation was given in terms of the covariance of the stationary structural responses. The objective function for the optimization was defined as the trace of the product of a positive semidefinite symmetric and the covariance of the stationary response. The covariances were obtained by solving a reduced-order Lyapunov equation. The objective function was shown to be general enough to represent displacement, interstory drifts, velocities, and accelerations at one or many points. A volume constraint was imposed to limit the design space, and the design variables were chosen as the relative densities in each element, which were bounded to achieve physically meaningful solutions. The material properties for intermediate densities were obtained using the SIMP interpolation rule; a linear hat projection filter was used to avoid numerical instabilities. The sensitivities of the performance function were obtained using an adjoint method, which requires the solution of an adjoint Lyapunov equation, also solved using the Lyapunov equation solver. Iterations were carried out using a gradient-based approach commonly employed in the topology optimization field.

The proposed framework was illustrated by conducting topology optimization of the lateral resisting system of a 9-story and 5-bay building. The design was performed for a set of parameters given by the benchmark problem. The domain was meshed using 180×144 Q4 elements, 864 frame elements in the lateral boundaries, and additional lumped floor masses, representing the lateral resisting system of a mid-rise building under stochastic earthquake loading. The goal was to minimize the maximum interstory drift among all floors. Two types of floor diaphragms were considered: rigid and flexible, which heavily influence the resulting topologies. The design consisted of a short beam spanning multiple floors with internal and external bracing patterns restraining the lumped masses for the flexible diaphragm case and two super braces with additional sets of braces for the rigid diaphragm case. In all cases, a uniform distribution of interstory drift variance was obtained. These results demonstrate the efficacy of the proposed approach for multi-objective topology optimization of stochastically excited structures.

REFERENCES

Bendsøe, M. P., & Kikuchi, N. (1988). "Generating optimal topologies in structural design using a homogenization method," *Computer Methods in Applied Mechanics and Engineering*, ***71***(2), 197–224.

Bendsøe, M. P., & Sigmund, O. (2003). *Topology optimization: theory, methods, and applications*. Springer.

Clough, R. W., & Penzien, J. (1993). *Dynamics of structures*, 2nd ed. New York McGraw-Hill.

Díaz, A., & Sigmund, O. (1995). "Checkerboard patterns in layout optimization," *Structural Optimization*, ***10***(1), 40–45.

Gomez, F., & Spencer, B. F. (2019a). "Topology optimization framework for structures subjected to stationary stochastic dynamic loads," *Structural and Multidisciplinary Optimization*, **59**(3), 813–833.

Gomez, F., & Spencer, B. F. (2019b). “Topology optimization of buildings subjected to stochastic base excitation,” *In Press*.

Lai, S. P. (1982). “Statistical characterization of strong ground motions using power spectral density function,” *Bulleting of the Seismological Society of America*, ***72*, **259-274.

Ma, Z.-D., Kikuchi, N., & Cheng, H.-C. (1995). “Topological design for vibrating structures,” *Computer Methods in Applied Mechanics and Engineering*, ***121***(1–4), 259–280.

Ohtori, Y., R. E. Christenson, B. F. Spencer, & S. J. Dyke. (2004). “Benchmark Control Problems for Seismically Excited Nonlinear Buildings,” *Journal of Engineering Mechanics*, **130**(4), 366–385.

Olhoff, N. (1989). “Multicriterion structural optimization via bound formulation and mathematical programming,” *Structural Optimization*, ***1***(1), 11–17.

Sigmund, O. (2007). “Morphology-based black and white filters for topology optimization,” *Structural and Multidisciplinary Optimization*, ***33***(4–5), 401–424.

Sigmund, O., & Petersson, J. (1998). “Numerical instabilities in topology optimization: A survey on procedures dealing with checkerboards, mesh-dependencies and local minima,” *Structural Optimization*, ***16***(1), 68–75.

Soong, T. T., & Grigoriu, M. (1993). *Random vibration of mechanical and structural systems*. PTR Prentice Hall.

Stromberg, L. L., Beghini, A., Baker, W. F., & Paulino, G. H. (2012). “Topology optimization for braced frames: Combining continuum and beam/column elements,” *Engineering Structures*, ***37***, 106–124.

Svanberg, K. (1987). “The method of moving asymptotes—a new method for structural optimization,” *International Journal for Numerical Methods in Engineering*, ***24***(2), 359–373.

Talischi, C., Paulino, G. H., Pereira, A., & Menezes, I. F. M. (2012). “PolyTop: a Matlab implementation of a general topology optimization framework using unstructured polygonal finite element meshes,” *Structural and Multidisciplinary Optimization*, ***45***(3), 329–357.

Xu, J., Spencer, B. F., Lu, X., Chen, X., & Lu, L. (2017). “Optimization of structures subject to stochastic dynamic loading,” *Computer-Aided Civil and Infrastructure Engineering*, **32**(8), 657–673.